Progress in Nonlinear Differential Equations and Their Applications
Volume 56

Victor A. Galaktionov
Juan Luis Vázquez

A Stability Technique for Evolution Partial Differential Equations

A Dynamical Systems Approach

Birkhäuser
Boston • Basel • Berlin

Victor A. Galaktionov
Department of Mathematical Sciences
University of Bath
Bath BA2 7AY
United Kingdon
and
Keldysh Institute of Applied Mathematics
Miusskaya Sq., 4
125047 Moscow, Russia

Juan Luis Vázquez
Deparment of Mathematics
Universidad Autónoma de Madrid
238049 Madrid
Spain

Library of Congress Cataloging-in-Publication Data

A CIP catalogue record for this book is available from the Library of Congress, Washington D.C., USA

AMS Subject Classifications: Primary: 35-XX, 35D, 35K, 35K55, 35K65; Secondary: 37-XX, 37L, 35Bxx, 35Qxx, 37N10, 34-XX, 34Dxx, 34Exx, 34Gxx, 76-XX, 80-XX, 80A20, 80A22, 80A23, 80A25

ISBN-13: 978-1-4612-7396-7 e-ISBN-13: 978-1-4612-2050-3
DOI: 10.1007/978-1-4612-2050-3

Printed on acid-free paper.

Birkhäuser

Softcover reprint of the hardcover 1st edition 2004

9 8 7 6 5 4 3 2 1 SPIN 10733778

Birkhäuser is part of *Springer Science+Business Media*

www.birkhauser.com

This book is dedicated to our wives, Olga and Mariluz,
and to our children, Oleg, Isabel, and Miguel.

Contents

Introduction: Stability Approach and Nonlinear Models

The S-Theorem

This book contains the description and application of a method of asymptotic analysis, a new stability theorem that we call the *S-Theorem*, originated in the study of the large-time behaviour of a class of partial differential equations known generally as *nonlinear reaction-diffusion equations.* These equations are among the best-known equations of mathematical physics. But, as shown in the text, the method has a more general scope in the study of evolution problems which can be posed in an abstract setting as infinite-dimensional dynamical systems. This is why we often refer to it as a *Dynamical Systems Approach.*

The study of asymptotic behaviour of solutions of evolution equations is a classical subject of mechanics and dynamical systems, and a number of quite effective methods have been developed, such as Lyapunov techniques, stable and centre manifold analysis, scaling and renormalization group arguments, etc. These methods can be used quite successfully to understand the asymptotic properties of many quasilinear reaction-diffusion equations, also known as nonlinear heat equations, in particular, when they admit global-in-time solutions, so that no essential singularities occur in the large-time evolution. In principle, we will not deal with such problems with known global behaviour, and will be concerned with problems that exhibit a complicated structure of asymptotic patterns that makes our analysis necessary or convenient.

The method presented here is suitable for application to different evolution problems described by nonlinear partial differential equations (PDEs) of parabolic or hyperbolic type, involving first-order, second-order or higher-order operators, many of them admitting free boundaries, or for other types of equations or systems. The common feature is that these evolution problems can be formulated as *asymptotically small perturbations* of certain dynamical systems with better-known behaviour. Now, it usually happens that the perturbation is small in a very weak sense, hence the difficulty (or impossibility) of applying more classical techniques.

Though the method originated with the analysis of critical behaviour for evolution PDEs, in its abstract formulation it deals with a nonautonomous abstract differential equation (NDE)

$$u_t = \mathbf{A}(u) + \mathbf{C}(u,t), \quad t > 0, \tag{1}$$

where u has values in a Banach space, like an L^p space, $\mathbf{A}$ is an autonomous (time-independent) operator and $\mathbf{C}$ is an asymptotically small perturbation, so that $\mathbf{C}(u(t),t) \to 0$ as $t \to \infty$ along orbits $\{u(t)\}$ of the evolution in a sense to be made precise, which in practice can be quite weak. We work in a situation in which the autonomous (limit) differential equation (ADE)

$$u_t = \mathbf{A}(u) \tag{2}$$

has a well-known asymptotic behaviour, and we want to prove that for large times the orbits of the original evolution problem converge to a certain class of limits of the autonomous equation.

More precisely, we want to prove that the orbits of (NDE) are attracted by a certain limit set Ω_* of (ADE), which may consist of equilibria of the autonomous equation, or it can be a more complicated object. A set of *three basic requirements* allows this conclusion, the main one being the Lyapunov stability of the closed set Ω_*, and this is the contents of the S-Theorem. It is typical of standard methods that such stability assumptions have to be imposed on the original equation (NDE). An important feature of our method is that it places the stability assumption on the limit equation (ADE). Note also that the convergence result *does not* depend on the knowledge of any rate of decay for the perturbation $\mathbf{C}(u,t)$ as t grows.

In Chapter 1 we state our main stability theorem (S-Theorem, in short). We establish that under three hypotheses (H1)–(H3), the omega-limit set of a perturbed dynamical system is stable under arbitrary asymptotically small perturbation. This result will be used throughout the book. The problem has been formulated above for convenience in the language of differential equations, but actually the S-Theorem is of a more general character, and applies to abstract dynamical systems posed in a complete metric space.

Asymptotics of nonlinear evolution PDEs

The rest of the book is devoted to the study of a selection of nonlinear asymptotic phenomena which occur for classes of equations involving different nonlinear operators. Indeed, the second goal of the book is to contribute a number of techniques and results to the wide field of *asymptotics of nonlinear evolution PDEs*.

The concrete examples of application have been chosen because they are relevant asymptotic problems that attracted the interest of the authors, were not covered by existing theories, and motivated the development of this theory. We present nine main examples, starting with classical reaction-diffusion-convection theory, and go on to cover subjects in blow-up, fluid flows (Navier–Stokes), Hamilton–Jacobi and fully nonlinear equations. We contribute to the theory of such equations, describe some general nonlinear effects and present a classification of the involved singularities.

Indeed, a first motivation of the theory has been the study of typical models of nonlinear diffusion. We devote Chapter 2 to presenting the main equations along with the concepts, tools and typical results on existence, uniqueness and differential properties of weak solutions, that might be useful in setting the context, as a technical preliminary for subsequent chapters. We will in particular examine the known *asymptotic properties* as $t \to \infty$. We demonstrate basic mathematical tools developed in the second half of the twentieth century on a benchmark equation, the *Porous Medium Equation* (PME, in short)

$$u_t = \Delta u^m \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+, \tag{3}$$

where $m > 1$ is a fixed exponent. For $m = 1$ it is just the classical *Heat Equation*.

In subsequent chapters, our text contributes to the general theory by supplying a further analysis tool that has allowed the authors to perform a complete asymptotic study in a number of open cases, many of them involving critical situations and striking phenomena of singularity formation. Especially, we will be interested in *blow-up properties*, when solutions become unbounded (in L^∞ or in another natural norm) after a finite time.

Before we proceed with the outline of the applications, let us try to understand in a few words why the study of nonlinear evolution equations or reaction-diffusion type leads to the consideration of *small asymptotic perturbations* of better-known autonomous dynamical systems.

Consider the case of critical diffusion-absorption treated in Chapter 4. It is well known that the solutions of the heat equation $u_t = \Delta u$ and the PME (3) posed in the whole space $\mathbb{R}^N$ with integrable initial data $u_0 \in L^1(\mathbb{R}^N)$, decay as $t \to \infty$ like $O(t^{-\alpha})$ for an exponent α that is shown to be $\alpha = N/[N(m-1)+2]$.

When we want to be more precise we rescale (i.e., we zoom) the variable u into a new variable θ that equals u times the decay factor t^α, hence it has size $O(1)$ for large t. But if we want θ to be a solution of a nice equation we have to also re-scale space in the form $x = \xi\, t^{\alpha/N}$. We are also interested for the same reason in using logarithmic time $\tau = \ln t$. This is all well known using dimensional analysis and exploits the property of scale invariance of the equation, and leads to the rescaled PME for $\theta(\xi,\tau) = t^\alpha\, u(x,t)$:

$$\theta_\tau = \mathbf{A}(\theta) \equiv \Delta\theta^m + \tfrac{\alpha}{N}\,\xi \cdot \nabla\theta + \alpha\,\theta. \tag{4}$$

It is an autonomous equation and its solutions tend to a nontrivial equilibrium, namely, the Gaussian kernel if $m = 1$, and the ZKB profile if $m > 1$. The asymptotic profile of the original problem is now read as the transformation of that equilibrium in terms of u.

Suppose now that you consider the more complicated model equation

$$u_t = \Delta u^m - u^\beta, \tag{5}$$

with $\beta, m \geq 1$. This is a model of nonlinear diffusion in an absorptive medium, well known in the literature. The absorption term is not an asymptotically small perturbation in principle. Now, we happen to know that the decay rate for this equation is the same as before when $\beta > \beta_* = m + 2/N$. If this is so we perform the same type of re-scaling to find

$$\theta_\tau = \mathbf{A}(\theta) + \mathbf{C}(\theta, \tau), \qquad \mathbf{C}(\theta, \tau) = -e^{-\sigma\,\tau}\theta^\beta, \qquad \sigma = (\beta - \beta_*)\alpha. \tag{6}$$

In this form we arrive at an asymptotically small perturbation of the rescaled PME (4) and the problem falls into the scope of the text. The appearance of the small exponential factor reminds us that we have lost the scale invariance in the original equation (5). Curiously, the most difficult analysis occurs for the critical case $\beta = \beta_*$, where we will concentrate the attention, and is naturally done with the S-Theorem.

Description of the applications

In Chapter 3 we perform a first application of the S-Theorem to study the asymptotic behaviour of nonnegative solutions for the equation of *superslow diffusion* which in N-dimensional geometry takes the form

$$u_t = \Delta(e^{-1/u}). \tag{7}$$

It can be treated as a formal limit case of the PME with $m = \infty$. We separately consider the initial-value problem for $t > 0$ in a bounded domain $\Omega \subset \mathbb{R}^N$ and the Cauchy problem in $\mathbb{R} \times \mathbb{R}_+$. Interesting transformations are needed to present those problems as small asymptotic perturbations of some well-known equation, and this is an important aspect of the theory. It turns out that in these two problems the asymptotic patterns look similar, but the rescaled variables and perturbed equations differ essentially. In the case of the bounded domain the rescaled equation with small asymptotic perturbations is rather involved and is given by

$$\begin{aligned}\theta_\tau &= \mathbf{A}(\theta) + \frac{4\,\ln\tau}{\tau}\theta\,\Delta\theta + \frac{2}{\tau}\,(\theta - \theta\ln\theta\,\Delta\theta)\\ &\quad + \frac{4\ln^2\tau}{\tau^2}\theta\,\Delta\theta - \frac{4\,\ln\tau}{\tau^2}\theta\ln\theta\,\Delta\theta + \tfrac{1}{\tau^2}\theta\,(\ln\theta)^2\,\Delta\theta,\end{aligned}$$

with $\mathbf{A}(\theta) = \theta\,\Delta\theta + \theta$.

In Chapter 4 we describe the asymptotic behaviour of a PME with absorption in the case of a *critical* exponent,

$$u_t = \Delta u^m - u^\beta \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+, \quad \beta = \beta_* = m + 2/N. \tag{8}$$

The exponent β_* (often called *critical Fujita exponent* for equations with source term $+u^\beta$) has been chosen because it is precisely the case when more standard methods of asymptotic analysis fail. Briefly explained, the difficulty stems from the fact that the two operators on the right-hand side have effects of the same order of magnitude, as can be easily shown by dimensional analysis or scaling. Thus, in the rescaling calculations done above for $\beta > \beta_*$, we see that the perturbation is not small when we pass to the limit $\beta \to \beta_*$. Consequently, the problem exhibits a typical *critical* situation, which is called a *resonance* in physical parlance. One of the main consequences is that the decay rate is modified to include extra *logarithmic factors* (a typical feature of resonance in dynamical systems).

The authors used the S-Theorem in 1991 to prove that all weak, space-integrable solutions behave for $t \to \infty$ as a unique orbit of the PME *without absorption*, and the resonance is felt as a rescaling in u and x by slow-growth unbounded factors, logarithmic functions of time. This is an example of a *transitional* behaviour between two different asymptotic structures for $\beta < \beta_*$ and $\beta > \beta_*$. The behaviour for the critical exponent $\beta = \beta_*$ then inherits certain features of both the subcritical and the supercritical ranges. This kind of transitional behaviour has a quite general nature and occurs for other equations; we will present some other instances of the phenomenon. The paper [169] was the first instance of an application of the "dynamical systems approach with asymptotically small perturbations" developed in this book.

Chapter 5 deals with the asymptotics of a problem involving extinction. *Extinction in finite time* is the term which denotes the phenomenon whereby a positive solution of an evolution process becomes identically zero after a finite time T, $u(\cdot, T) = 0$. The phenomenon is also called complete *quenching*. It is well known that this is not possible for the standard problems associated to the heat equation and other parabolic evolution operators with good coefficients. The phenomenon arises in nonlinear equations due to the presence of terms that either degenerate or are singular at $u = 0$. The extinction of a solution is usually associated with the formation of a singularity for the solution at the level of some derivative. Therefore, it can be understood as blow-up for the derivatives of the solution, with the advantage that the L^∞ norm of the solution itself remains bounded. In this chapter we still consider the PME with absorption, but the presence of a strong absorption term produces extinction. We concentrate on the equation with another *critical* exponent

$$u_t = \Delta u^m - u^p, \quad m > 1, \quad p = p_* = 2 - m < 1. \tag{9}$$

In this case the singular behaviour close to the extinction time, $t \to T < \infty$, is governed by the ODE without diffusion:

$$u_t = -u^{2-m}.$$

This is the first time that we face the case of *singular perturbation*: the limit equation is of lower order than the original PME with absorption. As is well known from the theory of singular perturbations, the passage to the limit becomes a hard problem. In order to apply the S-Theorem, we need to prove several estimates on rescaled orbits in a metric space C_ρ with a singular weight.

We follow with two chapters where the S-Theorem is used in combination with the technique of *Matched Asymptotic Expansions.* This is a very important tool of asymptotic analysis that is needed to reflect the multiple behaviour of many problems arising in several applied fields, hence our interest in the study that combines both machineries. Chapter 6 is devoted to the study of the fast diffusion equation with *critical* parameter

$$u_t = \Delta u^m \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+, \quad m = m_* = (N-2)/N, \quad N \geq 3. \tag{10}$$

We establish that $m = m_*$ corresponds to the transition between two different types of self-similar asymptotic behaviour in a neighbourhood of the critical value for $m > m_*$ (self-similarity of the *first kind* given by the ZKB solution), and $0 < m < m_*$ (self-similarity of the *second kind*). As a consequence, we describe two different asymptotic domains, the outer and the inner ones, with quite different asymptotic scalings. The leading part of the asymptotics in the outer domain is governed by a radial solution of the first-order equation (the conservation law)

$$v_t + N(v^{(N-2)/N})_s = 0, \quad \text{where } s = \ln|x|,$$

to which the stability theory applies. The inner one has a simple "flat" shape and some parabolic properties are necessary to match both the asymptotics.

Chapter 7 is devoted to the PME in exterior domains. We need to use expansions in the inner and outer regions and a matching procedure (the approach is different from that in Chapter 6). The main feature of the topic is the role played by singular solutions as asymptotic limits in the S-Theorem. We address here the critical situation that appears in dimension two and produces a typical $\ln(t)$ factor in the delicate matching process.

We cover next some topics from *fluid mechanics*. In Chapter 8 we turn to a classical problem and study a singularly perturbed dynamical system which describes some special blow-up patterns of the Navier–Stokes equations in $\mathbb{R}^2$,

$$\begin{cases} u_t + uu_x + vu_y = -p_x/\rho + \nu\Delta u, \\ v_t + uv_x + vv_y = -p_y/\rho + \nu\Delta v, \\ u_x + v_y = 0, \end{cases} \tag{11}$$

where (u, v) is the velocity field, p is the pressure, $\rho > 0$ is the constant density and $\nu > 0$ is the constant kinematic viscosity. We are interested in the particular solutions similar to the famous stationary *von Kármán solution* of the form

$$u = \int_0^x f(z,t)dz, \quad v = -yf(x,t), \quad p = h(x,t).$$

They describe a *plane jet* with a stagnation point at (0, 0) and free boundaries. Then the function f solves a *semilinear nonlocal heat equation*

$$f_t + \left(\int_0^x f(z,t)dz\right) f_x - f^2 = \nu f_{xx}$$

with free boundary conditions. We study the first stable blow-up pattern which gives the asymptotic structure of the plane jet for the Navier–Stokes equations. In particular, we prove that asymptotically this generic blow-up pattern is described by a nonlocal semilinear first-order Hamilton–Jacobi equation

$$f_t + \left(\int_0^x f(z,t)dz \right) f_x - f^2 = 0,$$

so that this asymptotic analysis falls in the scope of a singular perturbation theory.

In Chapter 9 we study a problem of *blow-up*, i.e., the solutions become unbounded in a finite time, and the profile that is formed at this time is under investigation. Blow-up is a major area of research in nonlinear evolution equations, cf. [32, 180, 286]. We consider the semilinear equation with "almost linear" reaction term

$$u_t = u_{xx} + (1+u)\ln^2(1+u) \quad \text{in } \mathbb{R} \times \mathbb{R}_+. \tag{12}$$

The study presents an important aspect, i.e., the asymptotic *degeneracy* of the parabolic equations near blow-up. More concretely, we prove that for bounded bell-shaped initial data $u_0(x) \geq 0$, the asymptotic behaviour as $t \to T$ is described by the nonlinear quadratic *Hamilton–Jacobi equation*

$$u_t = \frac{(u_x)^2}{1+u} + (1+u)\ln^2(1+u),$$

and the S-Theorem makes it possible to pass to the limit in a singularly perturbed dynamical system. Finally we prove that this equation exhibits *regional* blow-up where the *blow-up set* for bell-shaped data has a finite length equal to 2π. We also study periodic blow-up patterns and their localization. This work was developed in the paper [173], written in 1991, and was a major source of inspiration in developing the idea of *reduced omega-limit sets*, an important ingredient in the sharp formulation of the S-Theorem.

In Chapter 10 we present a general theory of such degeneracy effect of convergence to Hamilton–Jacobi solutions. It applies to a class of quasilinear equations with different types of blow-up, such as *single-point*, regional or *global* blow-up. As a basic model, we classify the asymptotics of the quasilinear heat equation

$$u_t = \nabla \cdot (\ln^\sigma(1+u)\nabla u) + (1+u)(\ln(1+u))^{\beta(\sigma+1)-\sigma} \tag{13}$$

for different values of the parameters $\sigma \geq 0$ and $\beta > 1$. It is important that this equation describes all three types of blow-up: (i) regional for $\beta = 2$, (ii) *single-point* for $\beta > 2$ and (iii) *global* if $\beta \in (1,2)$. The asymptotic blow-up patterns are proved to have different space-time structures in these three cases.

We perform in Chapter 11 the asymptotic analysis of a *fully nonlinear* parabolic equation from detonation theory. The parabolic equation

$$u_t + \tfrac{1}{2}(u_x)^2 = f(cuu_{xx}) + \ln u \quad (c > 0) \tag{14}$$

with a smooth strictly monotone increasing function, $f(s) = \ln((e^s - 1)/s)$, describes unstability of the square Zel'dovich–von Neuman–Doering (ZND) wave in detonation theory. The model is due to Buckmaster and Ludford. We study the finite time *quenching* behaviour as $t \to T$ when an initially strictly positive solution touches the singular level $u = 0$, where the diffusion-like operator degenerates and the absorption term $\ln u$ becomes singular. We establish that this behaviour is described by a singularly perturbed linear first-order equation of Hamilton–Jacobi type. It is important that the solution does not admit any proper continuation beyond quenching time, for $t > T$. This means complete collapse of the ZND-wave at the quenching point.

We add a last Chapter 12, where we briefly describe further, sometimes not very straightforward, extensions and generalizations, and give a list of related references. We show how to extend our dynamical system approach by using an extra topological structure in the metric space and hence modifying the notion of the uniform Lyapunov stability. Under a suitable assumption on the corresponding topological structure of the reduced omega-limit set of the autonomous equation, we then obtain more detailed description of the omega-limits of a class of individual orbits. Another new application is time-dependent *homogenization*-like problems for the PME or other parabolic equations with highly oscillatory coefficients.

We also demonstrate that the S-Theorem exhibits natural applications to a number of problems for *higher-order parabolic equations* with reaction/absorption-like terms, and as typical examples we treat the semilinear $2m^{\text{th}}$-order equations

$$u_t = -(-\Delta)^m u \pm |u|^{p-1} u \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+ \tag{15}$$

with integer $m > 1$ and exponent $p > 1$, which induce typical examples of semigroups without order-preserving properties (available for $m = 1$ only via the *Maximum Principle*).

Summing up, the nonlinear models described above play the role of key examples in explaining some crucial distinctive features of the applications of the stability theorem (Chapter 1) to a class of similar perturbed dynamical systems. Of course, such an analysis admits various extensions and generalizations to wide classes of problems, where a similar kind of perturbations occurs. We describe such generalizations in Remarks at the end of each chapter.

The equations and problems we deal with were mostly well known and were actively studied from different points of view in the last two decades in the framework of the growing theory of nonlinear partial differential equations, and the questions of (local-in-time) existence, uniqueness and regularity of solutions are documented in the literature. We present suitable references in the final section (remarks and comments on the literature) of each chapter. Though we have selected applications involving nonlinear heat equations, the abstract stability theory, on which the analysis relies, has a wider scope, and some of the examples are directed to promote such extension.

This book presents a unified approach to the study of the asymptotic behaviour of several classes of nonlinear equations. The main results were obtained by the authors during the last twelve years. These classes of asymptotic problems for nonautonomous dynamical systems were not discussed in monographs on the theory of nonlinear PDEs.

Prerequisites and use

The book assumes some knowledge of the fundamentals of partial differential equations, ordinary differential equations, and functional analysis. A certain exposure to dynamical systems will be helpful as background to understand the main result and the general philosophy. The examples of application which form the bulk of the book assume some knowledge of the main topics of nonlinear partial differential equations of evolution type and their asymptotics, e.g., global or local well-posedness and Lyapunov techniques. It is not an absolute prerequisite to read our corresponding introductory text but it explains the context and why the present method is useful. Much of the necessary material on basic theory and asymptotics of nonlinear heat equations is summarized in Chapter 2, where further references are given. More general references are [293] and [286], which deals in great detail with blow-up problems. Explanations, references and hints will be given as the text proceeds.

The book is meant for an advanced graduate level and can be taught to students in mathematics and physics interested in evolution equations and asymptotics in one semester if a proper selection of the topics is made. It can be combined with standard evolution equations and asymptotics topics into a whole year in various ways. The whole text could serve as a reference work on the S-Theorem and its applications.

Acknowledgments

The authors are especially thankful to M. Chaves, S. Gerbi, R. Kersner, L.A. Peletier, S.A. Posashkov and F. Quirós, the co-authors of some of the papers we have used in the presentation of the results. We also thank R. Ferreira, always helpful with the numerics and graphics. During the last decade, both authors had a great opportunity to talk about nonlinear equations and mathematical methods with many experts in mechanics, applied mathematics and theory of nonlinear PDEs. We would like to thank the colleagues and friends who were involved at different times in discussing with us these issues, among them: G.I. Barenblatt, J. Bebernes, A. Bressan, H. Brezis, M. Fila, J.R. King, S. Kamin, the late S.N. Kruzhkov, H.A. Levine, S.I. Pohozaev, S.R. Svirshchevskii, L. Véron and E. Zuazua.

Most of the results included in this book were established when the first author spent his sabbatical years as Professor Visitante at the Departamento de Matemáticas, Universidad Autónoma de Madrid, in 1992–95. During the last years he was also supported by Fundación Iberdrola. Both authors are thankful to these institutions for their support. The first author was also encouraged by the Department of Mathematical Sciences, University of Bath, which always supported his collaboration with the PDEs School in the Universidad Autónoma.

Victor A. Galaktionov

Juan Luis Vázquez

Bath and Madrid, December 2002

1

Stability Theorem: A Dynamical Systems Approach

This chapter contains the statement and proof of the abstract stability result on which the theory of later chapters relies. The large-time behaviour for different PDEs of evolution type is seen in an abstract setting from the unifying point of view of dynamical systems posed in an arbitrary metric space.

1.1 Perturbed dynamical systems

We are interested in describing the asymptotic behaviour of different evolution processes that offer difficulties when treated by standard methods. In all of them we arrive after suitable transformations at a general formulation in the form of a *nonautonomous* dynamical system

$$u_t = \mathbf{B}(u,t) \equiv \mathbf{A}(u) + \mathbf{C}(u,t), \quad t > 0; \quad u(0) = u_0,$$

where $\mathbf{A}$ is an autonomous operator and $\mathbf{C}$ is an asymptotically small perturbation, so that along a typical solution $u(t)$, there holds $\mathbf{C}(u(t),t) \to 0$ as $t \to \infty$ in some (possibly weak) sense. We want to prove that for large times the orbits of the original evolution problem converge to a certain set of limits of the orbits of the autonomous equation. In order to make this statement precise, we define the *ω-limit set* of the given orbit, $\omega(u)$, and we select a certain subset Ω_* of the global ω-limit set of the *autonomous* equation

$$u_t = \mathbf{A}(u).$$

In this notation we want to prove that

$$\omega(u) \subseteq \Omega_*.$$

The precise result demands carefully stated assumptions and conclusions which make up the main stability theorem.

Results on asymptotic convergence of solutions of evolution equations are obtained under suitable assumptions on the character of the evolution. Typically such

V. A. Galaktionov et al., *A Stability Technique for Evolution Partial Differential Equations*

assumptions concern the original equation under consideration. An important feature of our method is that it places the main assumption on the limit equation: it consists of the hypothesis of *uniform stability* in the Lyapunov sense on the set Ω_* with respect to the flow generated by the autonomous equation. In some applications this is asking too much, but a convenient modification works: we need only establish the stability of a certain *reduced omega-limit set*. It is important that the convergence result does *not* depend on any information about the rate of decay of the perturbation in the equation.

Finally, let us remark before we proceed that we formulate the problem using the language of abstract differential equations, but this has to be understood as a convenient way of presenting the result in view of the typical applications. Actually, the general result deals with the *large-time behaviour of classes of curves defined in a metric space* which enjoy certain properties, and no differentiation is essential in the statements or arguments. In the applications we will use the fact that the curves under consideration are solutions of differential equations to make sure that the needed properties hold.

1.2 Some concepts from dynamical systems

We will be working in this book with solutions of differential equations which can be viewed as continuous curves, $u \in C(I : X)$, with values in a complete metric space X. Typically X will be an L^p-space or other function space but this chapter will make no such restriction. We denote by $d(\cdot, \cdot)$ the metric in X. I can be the real line but it is usually an infinite interval of the form $I = [t_0, \infty)$ (a forward half-line), and $t_0 = t_0(u)$ may depend upon the curve under consideration. Curves which are solutions of an evolution process are often referred to as *trajectories* of the process or *orbits*, though the last name usually refers to the image of the curve, cf. [188]. Thus, for any curve u with domain $\mathbb{R}$ we define the *complete orbit* as

$$\gamma(u) = \{u(s) : s \in \mathbb{R}\} \subset X,$$

and the complete trajectory as the complete curve, which is identified with the set $\Gamma(u) = \{(s, u(s)) : s \in \mathbb{R}\} \subset \mathbb{R} \times X$. We are mainly interested in forward orbits of curves defined in a half-line. The *forward orbit* starting at time $t \geq t_0(u)$ is defined as

$$\gamma^+(u, t) = \{u(s) : s \geq t\}. \tag{1.1}$$

If $t = 0$ we drop the t, $\gamma^+(u) = \gamma^+(u, 0)$. We include the case $t < t_0$, when we write $\gamma^+(u, t) = \gamma^+(u, t_0)$; this may seem unnecessary but is convenient in writing general statements. Usually, the solution of an evolution process (i.e., a differential equation) is uniquely determined by its initial data and then it is convenient to use the notation $\gamma^+(u, t) = \gamma^+(u_0, t)$, where $u(t_0) = u_0 \in X$, but uniqueness in terms of the initial data is not a requirement for what follows.

The *ω-limit set* of a curve $u : I \to X$ is defined as

$$\omega(u) = \{f \in X : \exists \text{ a sequence } \{t_j\} \to \infty \text{ such that } u(t_j) \to f\}. \quad (1.2)$$

It is a subset of X that can also be written as

$$\omega(u) = \bigcap_{t \geq t_0} \text{clos}(\gamma^+(u, t)),$$

where $\text{clos}(E)$ denotes the closure of a set E in X. As before, for solutions of differential equations, we also write the omega limit in terms of the initial data, $\omega(u) = \omega(u_0)$. The following result is well known, cf. [189], [297].

Lemma 1.1 *$\omega(u)$ is a closed subset of X. If $\gamma^+(u, t)$ is relatively compact, then $\omega(u)$ is nonempty, connected and compact.*

Generalizing the previous definition, if we have a family of forward curves $\mathcal{E}$ with values in the same metric space, we introduce its ω-limit set as

$$\omega(\mathcal{E}) = \bigcap_{t \geq t_0} \text{clos}(\bigcup_{u \in \mathcal{E}} \gamma^+(u, t)).$$

It can be alternatively described as follows: $\omega(\mathcal{E}) = \{f \in X : \exists$ a sequence $\{t_j\} \to \infty$ and a sequence of solutions $\{u_j\} \subset \mathcal{E}$ such that $u_j(t_j) \to f\}$. It must be observed that the ω-limit of a set is usually larger than the union of the ω-limits of its elements, $\bigcup\{\omega(u) : u \in \mathcal{E}\} \neq \omega(\mathcal{E})$, cf. example at the end of Section 1.5.

Families of solutions appear naturally in the study of differential equations as the solutions of an initial-value problem of the form

$$u_t = \mathbf{A}(u), \quad t > 0; \quad u(0) = u_0,$$

where $\mathbf{A}$ does not depend on time. Typically, the problem generates a *continuous semigroup*, i.e., a continuous map $S : X \times [0, \infty) \to X$, such that, if we write $S(t)x = S(x, t)$ as usual, the maps $S(t)$ satisfy

(i) $S(0)u_0 = u_0$ for every $u_0 \in X$,

(ii) $S(t + s)u_0 = S(t)S(s)u_0$ for every $u_0 \in X$ and $t, s \geq 0$.

In that case we can write the unique solution $u = u(t)$ with initial value u_0 as $u(t) = S(t)u_0$ for $t > 0$.

We need two further definitions. A set $E \subseteq X$ is called *(forward) invariant under* S if for every $t > 0$, we have $S(t)E \subseteq E$. A set F is said to *attract* a set E if $d(S(t)E, F) \to 0$ as $t \to \infty$.

Here is a typical result in this setting, where we write the ω-limit in terms of the initial values of the orbits, cf. [189].

Lemma 1.2 *(i) If $E \subset X$ is nonempty and its orbit $\gamma^+(E)$ is relatively compact, then $\omega(E)$ is nonempty, compact and attracts E.*
(ii) If $E \subset X$ is connected, then $\omega(E)$ is connected.
(iii) For any set $E \subset X$, for which $\omega(E)$ is compact and $\omega(E)$ attracts E, the set $\omega(E)$ is invariant.

The concepts of invariance and attraction can be immediately generalized to a family $\mathcal{L}$ of curves defined on a common interval, say $I = [0, \infty)$, with no relation to semigroups. We just define $S(t)(E) = \{u(t) : u \in \mathcal{L}, u(0) \in E\}$ for every set $E \subseteq \mathcal{L}(0) = \{u(0) : u \in \mathcal{L}\}$. Then, parts (i) and (ii) of the above result remain true, even if we do not necessarily have the semigroup hypothesis $S(t)S(s) = S(t+s)$. The invariance may not hold in this general setting.

1.3 The three hypotheses

Our asymptotic result can be formulated in a topological way with no reference to PDEs. It concerns the asymptotic properties of two families of curves $\mathcal{L}$ and $\mathcal{L}_*$ mapping forward-infinite intervals of the real line into a metric space (the same for both families). A set of three basic hypotheses are imposed on these families of curves or trajectories. These hypotheses are briefly summarized as *compactness, convergence and reduced stability*. Here is the detailed statement and preliminary analysis of them.

(H1) COMPACTNESS. We consider a class $\mathcal{L}$ of functions $u \in C([t_0, \infty) : X)$ defined for $t \geq t_0(u)$ with values in a complete metric space X. We assume that the corresponding orbits $\{u(t) : t \geq t_0\}$ are relatively compact in X. Moreover, we consider the trajectories after a time shift

$$u^{\tau}(t) = u(t+\tau), \quad t, \tau > 0,$$

and assume that the set of curves $\{u^{\tau}(t)\}$ with index $\tau > \tau_0$ is relatively compact in $L^{\infty}_{\mathrm{loc}}([t_0, \infty) : X)$.

It follows from (H1) that the forward orbit $\gamma^{+}(u, t)$ is relatively compact in X. Hence, the omega-limit $\omega(u)$ is nonempty and compact.

(H2) CONVERGENCE. $\mathcal{L}$ is a small asymptotic perturbation of $\mathcal{L}_*$ in the following sense: given a curve $u \in \mathcal{L}$, if for a sequence $\{t_j\} \to \infty$ the sequence $\{u(t_j + t)\}$ converges in $L^{\infty}_{\mathrm{loc}}([0, \infty) : X)$ as $j \to \infty$ to a function $v(t)$, then v belongs to $\mathcal{L}_*$.

Before we proceed further, we make some remarks and comments. Because of intended application, we think of $\mathcal{L}$ as a suitable family of solutions of an evolution process described (at least formally) by a nonautonomous abstract differential equation

$$u_t = \mathbf{B}(u, t), \quad t > 0. \tag{1.3}$$

This is to be compared for large times with the autonomous equation

$$v_t = \mathbf{A}(v), \quad t > 0, \tag{1.4}$$

more precisely, with a particular family of solutions $\mathcal{L}_*$ of the latter equation.

Assumption (H2) is our way of stating that (1.3) is an *asymptotically small perturbation* of (1.4), i.e., $\mathbf{B}(u, t)$ tends to $\mathbf{A}(u)$ as $t \to \infty$ in the very weak sense just described. Hence, (1.4) is called the *limit equation* and (1.3) is the *perturbed equation*. The curves v in $\mathcal{L}_*$ obtained in such a passage to the limit will be called *limit solutions*.

In order to make a difference between the two equations, we will use the standard dynamical systems notation for the solutions of the perturbed equation (1.3), and notations with stars for the limit equation (1.4). Thus, we write $\gamma_*^+(v,t)$, $\omega_*(v)$ and $\omega_*(\mathcal{E})$ for a solution $v \in \mathcal{L}_*$ and a set of solutions $\mathcal{E} \subset \mathcal{L}_*$, respectively.

Since hypothesis (H1) implies that $\gamma^+(u,t)$ lies in a relatively compact subset of X, it follows from (H2) that every limit solution $v(t)$ of u has a relatively compact forward orbit in X.

Proposition 1.3 *Given $u \in \mathcal{L}$, any limit solution $v \in \mathcal{L}_*$ can be defined for all $t \in \mathbb{R}$, i.e., it has a complete orbit. Moreover, for every $v_0 \in \omega(u)$, there is a limit solution v with initial data $v(0) = v_0$, and the complete orbit $\gamma_*(v)$ is contained in $\omega(u)$. Therefore, $\omega_*(v)$ is nonempty and compact.*

These facts need not be true for the whole class $\mathcal{L}_*$. In the typical applications to follow, both $\mathcal{L}$ and $\mathcal{L}_*$ are classes of weak or other generalized solutions on which weak regularity requirements are assumed. In accordance to this generality, the passage to the limit of hypothesis (H2) only imposes that the limit $v(t)$ of the sequence $\{u(t_j+t)\} \subseteq \mathcal{L}$ be a solution of the limit equation in $\mathcal{L}_*$, a condition that can often be obtained for generalized solutions of nonlinear heat equations under minimal or no estimates on the derivatives. Besides, no uniqueness result is implied up to the moment.

Let us now turn our attention to the third and main hypothesis. A main point in our result is that *no stability properties are assumed on* (1.3), but rather on its limit equation. We start by identifying the set where the omega limits of the solutions to equation (1.3) must lie. This is an important ingredient of the formulation. More precisely, we need to find a set $\Omega_* \subset X$ large enough to contain the *iterated omega limits*, by which we mean the ω_*-limits under (1.4) of the ω-limits of the perturbed equation (1.3). This means that

$$\Omega_* \supseteq \bigcup\{\omega_*(v) : \; v \in \mathcal{L}_*, \; v(0) \in \omega(u), \; u \in \mathcal{L}\}. \tag{1.5}$$

We can now formulate the last basic hypothesis in the strict form needed for the intended result to hold. Let $Y_0 = \bigcup\{\omega(u) : u \in \mathcal{L}\}$.

(H3) REDUCED UNIFORM STABILITY FOR EQUATION (1.4). We assume the existence of a closed subset Ω_* of X satisfying (1.5) which is *uniformly Y_0-stable* in the sense of Lyapunov: for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that if v is any curve in $\mathcal{L}_*$ with $v(0) \in Y_0$ and $d(v(0), \Omega_*) \leq \delta$, then

$$d(v(t), \Omega_*) \leq \varepsilon \quad \text{for every } t > 0.$$

Notice that we impose the stability of the set Ω_* with respect to perturbations in Y_0, thus the name *reduced stability*. It does not imply that the set Ω_* is invariant under the evolution defined by (1.4), but this will be true if we strengthen the stability condition by eliminating the requirement $v(0) \in Y_0$.

In line with comments already made, we point out that no special assumptions are made in the three hypotheses on the sense in which equations (1.4) and (1.3) are satisfied, or on other properties of the evolution. Thus, though it is usually true that (1.4) generates a semigroup in a metric space, such a property is of no concern for the general result we are aiming at. In particular, the class $\mathcal{L}$ may consist of only one solution. We will discuss later in Section 1.5 practical conditions under which the last hypothesis holds, less dependent on what $\omega(u)$ is.

1.4 The S-Theorem: Stability of omega-limit sets

We may now formulate and prove the announced main result, to be referred to in the book as the S-Theorem.

Theorem 1.4 *Under assumptions* (H1), (H2) *and* (H3), *the ω-limit set of any solution $u \in \mathcal{L}$ of the perturbed equation* (1.3) *is contained in Ω_*. In other words, each orbit of* (1.3) *is attracted by Ω_* as $t \to \infty$.*

Roughly speaking, the omega-limit set of the class of solutions $\mathcal{L}_*$ of the autonomous equation is stable under arbitrary perturbations of the equation which are asymptotically small in the sense defined above.

Proof. We divide the proof of the theorem into a series of lemmas. To begin with, for every fixed solution $u \in \mathcal{L}$ and every $\varepsilon > 0$, we define the *good* and *bad* sets

$$\mathcal{G}_\varepsilon = \{t > 0 : d(u(t), \Omega_*) \leq \varepsilon\}, \tag{1.6}$$

$$\mathcal{B}_\varepsilon = \{t > 0 : d(u(t), \Omega_*) > \varepsilon\}. \tag{1.7}$$

Clearly, $\mathcal{G}_\varepsilon \cup \mathcal{B}_\varepsilon = (0, \infty)$, $\mathcal{G}_\varepsilon \cap \mathcal{B}_\varepsilon = \emptyset$ and $\mathcal{B}_{\varepsilon_2} \subseteq \mathcal{B}_{\varepsilon_1}$ if $0 < \varepsilon_1 < \varepsilon_2$. The sets $\mathcal{B}_\varepsilon$ are open, the sets $\mathcal{G}_\varepsilon$ are closed. Of course, these sets depend on u which we take as fixed. We begin the study of $\mathcal{G}_\varepsilon$ and $\mathcal{B}_\varepsilon$ for large t with the following lemma:

Lemma 1.5 *For any $\varepsilon > 0$, there exists a sequence $\{T_j\} \to \infty$ contained in the set $\mathcal{G}_\varepsilon$.*

Proof. Let $\{t_j\} \to \infty$ be an arbitrary sequence. By (H1) the sequence of functions $\{u(t_j + s)\}$, is relatively compact; by (H2) we may assume (after passing to a subsequence if necessary) that as $j \to \infty$, $u(t_j + s)$ tends to $v(s)$, a function in $\mathcal{L}_*$ (i.e., a solution of (1.4)), uniformly on compact subintervals of $[0, \infty)$. Then $v(t) \in Y_0$ for all $t \geq 0$. Since Ω_* contains the omega limit of all solutions in $\mathcal{L}_*$ with data in Y_0, it follows from (H3) that $v(s)$ converges to Ω_* as $s \to \infty$, hence there exists $s_0 > 0$ such that for $s \geq s_0$,

$$d(v(s), \Omega_*) \leq \tfrac{\varepsilon}{2}. \tag{1.8}$$

The convergence of $u(t_j + s)$ to $v(s)$ implies that there exists j_0 such that

$$d(u(t_j + s), v(s)) \leq \tfrac{\varepsilon}{2} \tag{1.9}$$

for every $0 \leq s \leq s_0$ if $j \geq j_0$. Together, these inequalities imply

$$d(u(t_j + s_0), \Omega_*) \leq d(u(t_j + s_0), v(s_0)) + d(v(s_0), \Omega_*) \leq \varepsilon ,$$

which means that $\{t_j + s_0\}_{j \geq j_0} \subset \mathcal{G}_\varepsilon$. Now put $T_j = t_j + s_0$. □

We turn our attention to the set $\mathcal{B}_\varepsilon$. Since $\mathcal{B}_\varepsilon$ is an open set, if it is not empty it can be written as a countable or finite union of mutually disjoint open intervals

$$\mathcal{B}_\varepsilon = \bigcup_n I_n^\varepsilon , \qquad I_n^\varepsilon = (a_n^\varepsilon, b_n^\varepsilon), \tag{1.10}$$

with $0 < a_n^\varepsilon < b_n^\varepsilon$. Lemma 1.5 rules out the possibility of an unbounded interval going to $+\infty$. We have more.

Lemma 1.6 *The sequence of lengths* $\{l_n = b_n^\varepsilon - a_n^\varepsilon\}$ *is bounded;* $l_n \leq c = c(\varepsilon, u)$.

Proof. It is based on the same arguments as the proof of Lemma 1.5. Assuming that there exists a sequence $l_n \to \infty$ and that the intervals $(a_n^\varepsilon, b_n^\varepsilon)$ are ordered, we take $t_n = (a_n^\varepsilon + b_n^\varepsilon)/2$ and apply the previous argument to obtain an s_0 such that $t_j + s_0 \in \mathcal{G}_\varepsilon$ for a subsequence $\{t_j\}$. Since, by definition of I_n^ε, $t_j + t \notin \mathcal{G}_\varepsilon$ for any $t \in (0, l_j/2)$ and $l_j \to \infty$, we arrive at a contradiction. Therefore $\{l_n\}$ must be bounded. □

Finally, we prove that $\mathcal{B}_\varepsilon$ is empty or bounded. Figure 1.1 illustrates our analysis.

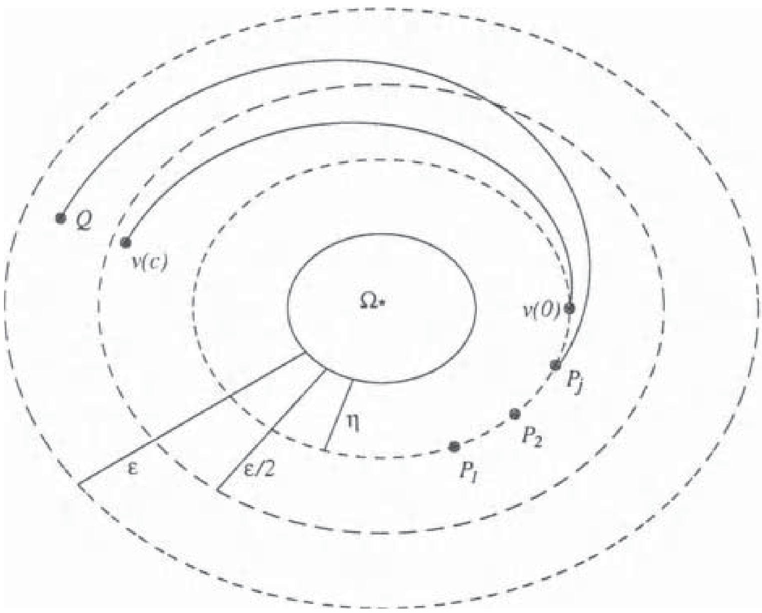

Fig. 1.1. Scheme for the proof of the S-Theorem. $P_j = u(t_j)$, $Q = u(t_j + c)$. Dashed lines bound neighbourhoods of Ω_*. Solid curves represent orbits of v and u starting from $t = 0$, $t = t_j$, resp.

Lemma 1.7 *There exists a constant $C = C(\varepsilon, u)$ such that*

$$(C, \infty) \subseteq \mathcal{G}_\varepsilon \,. \tag{1.11}$$

Proof. If (1.11) is false, there exists a subsequence $\{n_j\} \to \infty$ such that $I^\varepsilon_{n_j} \subset \mathcal{B}_\varepsilon$. After relabelling, we may assume that $n_j = j$ and that $a^\varepsilon_{j+1} > a^\varepsilon_j$. Let

$$\eta = \min\left\{\frac{\varepsilon}{4}, \tfrac{1}{2}\delta\left(\frac{\varepsilon}{2}\right)\right\}, \tag{1.12}$$

where δ is the function appearing in the definition of uniform stability of Ω_*, (H3). Since $\eta < \varepsilon$, we have $\mathcal{B}_\varepsilon \subseteq \mathcal{B}_\eta$ and to every interval I^ε_j, there corresponds an equal or larger interval $I^\eta_j = (a^\eta_j, b^\eta_j)$ contained in $\mathcal{B}_\eta$. Some of the intervals I^η_j may be repeated, though only a finite number of times each since

$$b^\eta_j - a^\eta_j \le c(\eta, u). \tag{1.13}$$

By (1.13) there will be an infinite number of different intervals left. From the definition of $\mathcal{B}_\eta$ it follows that if $t_j = a^\eta_j$, then

$$d(u(t_j), \Omega_*) = \eta \,. \tag{1.14}$$

Passing again to a subsequence that we still denote by $\{j\}$, the sequence of functions $\{u(t_j + t)\}$ converges uniformly on compact subintervals of $[0, \infty)$ to a solution $v(t) \in \mathcal{L}_*$. Therefore, for $j \ge j_0$, we have

$$d(v(0), u(t_j)) \le \eta \,, \tag{1.15}$$

so that from (1.14) $d(v(0), \Omega_*) \le 2\eta$ and by the stability hypothesis (H3)

$$d(v(t), \Omega_*) \le \tfrac{\varepsilon}{2} \tag{1.16}$$

for any $0 < t < \infty$. Now, the convergence of $u(t_j + t)$ towards $v(t)$ implies that given $c = c(\eta, u)$, there exists $j_1 > 0$ such that

$$d(u(t_j + t), v(t)) \le \tfrac{\varepsilon}{2} \tag{1.17}$$

for $0 \le t \le c(\eta, u)$ and $j \ge j_1$. Then

$$d(u(t_j + t), \Omega_*) \le d(u(t_j + t), v(t)) + d(v(t), \Omega_*) \le \varepsilon \,, \tag{1.18}$$

hence

$$[a^\varepsilon_j, b^\varepsilon_j] \subseteq [a^\eta_j, a^\eta_j + c(\eta, u)] \subseteq \mathcal{G}_\varepsilon$$

for all $j \ge j_1$, a contradiction with the definition of the intervals $(a_\varepsilon, b_\varepsilon)$. □

End of Proof of Theorem 1.4. By Lemma 1.7, for any solution $u \in \mathcal{L}$ of (1.3) and every $\varepsilon > 0$, there exists $t_1 = t_1(\varepsilon, u) > 0$ such that for $t \ge t_1$,

$$d(u(t), \Omega_*) \le \varepsilon.$$

It is then clear that the ω-limit set of the orbit $\{u(t)\}$ is contained in Ω_*. □

1.5 Practical stability assumptions

1.5.1 In the simplest formulation that we will find in some of the applications, we may take the set Ω_* as *the global omega-limit set* of the unperturbed dynamical system (1.4) in the class $\mathcal{L}_*$, defined as

$$\begin{aligned}\omega_*(\mathcal{L}_*) = \{f \in X : \exists \text{ a sequence } \{t_j\} \to \infty \text{ and a sequence} \\ \text{of solutions } \{v_j\} \subseteq \mathcal{L}_* \text{ such that } v_j(t_j) \to f\}.\end{aligned} \tag{1.19}$$

We may also replace the condition of reduced Y_0-stability by plain stability. We then have a stronger version of (H3) which reads

(H3a) UNIFORM STABILITY FOR EQUATION (1.4). We assume that the subset $\Omega_* = \omega_*(\mathcal{L}_*)$ is uniformly stable in the sense of Lyapunov: for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that if v is any solution of (1.4) in $\mathcal{L}_*$ such that $d(v(0), \Omega_*) \le \delta$, then

$$d(v(t), \Omega_*) \le \varepsilon \quad \text{for every } t > 0.$$

It is immediate from the stability property that Ω_* is invariant under the evolution defined by (1.4), i.e., for every solution v with initial data $v(0) \in \Omega_*$, we have $v(t) \in \Omega_*$ for every $t \ge 0$. In most cases studied below, Ω_* consists of stationary points, i.e., $v(t) = v(0)$ for all $t \ge 0$.

1.5.2 In many cases we will consider a smaller set than $\omega_*(\mathcal{L}_*)$. This happens for two reasons. Firstly, it is natural from the statement of the theorem that we look for a set as small as possible. Secondly, *the global ω_*-limit set may not be stable* in the sense of (H3). In practice, we will observe that *the ω-limits of the solutions of equation (1.3) have special properties* inherited in the limit from the solutions of equation (1.3), which is then seen as a kind of regularization of (1.4). In other words, there exists a certain subset $Y \subseteq X$ such that

$$Y \supseteq \bigcup \{\omega(u) : u \in \mathcal{L}\} = Y_0, \tag{1.20}$$

and the evolution is defined in $\mathcal{L}_*$ for all initial data $v(0) \in Y$. We then take

$$\Omega_* = \omega_*(Y). \tag{1.21}$$

The choice of Y will be of great importance in some of the applications. We call the set given by (1.21) the *reduced ω-limit set* of equation (1.4) relative to the subclass Y. If we impose hypothesis (H3) with Y replacing Y_0 and Ω_* as defined above, then Theorem 1.4 holds. In all cases considered in this work we have used equality as in formula (1.21), but inequality will also be acceptable: Ω_* must be closed and $\Omega_* \supseteq \omega_*(Y)$.

1.5.3 A further remark in the last situation concerns the convenience of checking condition (1.21) only on orbits, i.e., replacing it by

$$\Omega_* \supseteq \bigcup \{\omega(v) : v(0) \in Y\}, \tag{1.22}$$

which is sufficient for the theorem to hold. The following result shows that under certain conditions both concepts are equivalent.

Proposition 1.8 *Let $Y \subseteq X$ be a compact set and let S_* be a continuous dynamical system defined on Y with relatively compact orbits. If Ω_* is closed, stable and attracts all orbits of Y in the sense of* (1.22)*, then $\Omega_* \supseteq \omega_*(Y)$.*

Proof. Let x be the limit of a sequence $\{v_j(t_j)\}$ with $v_j(0) \in Y$ and $\{t_j\} \to \infty$. By compactness, $v_j(0)$ tends to some $y = w(0)$ after passing to a subsequence if necessary. By the attraction property, given $\delta > 0$, we have $d(w(t), \Omega_*) \leq \delta/2$ for all large $t \geq T$. On the other hand, for j large and by continuity we have $d(v_j(T), w(T)) \leq \delta/2$. It follows that $d(v_j(T), \Omega_*) \leq \delta$. But $v_j(T) \in Y$ is the initial data of the trajectory $\widetilde{v}$ defined by $\widetilde{v}(t) = v(t+T)$. Using now the stability, we conclude that for all large $j \geq j_0$,

$$d(v_j(t), \Omega_*) \leq \varepsilon \quad \text{for all } t \geq T,$$

which implies that $d(x, \Omega_*) \leq \varepsilon$ for every $\varepsilon > 0$, hence $x \in \Omega_*$. □

Let us remark that, when in the situation of Theorem 1.4 we take the minimal choice $Y = \omega(u)$, then for every $v \in \mathcal{L}_*$ with $v(0) \in Y$, we have $v(t) \in Y$, hence the evolution is defined in Y and the orbits are relatively compact. On the other hand, it is easy to construct simple examples of finite-dimensional dynamical systems where the result is false if Ω_* is not stable. For instance, we can construct a gradient flow in a ball Y in the plane and take as Ω_* the set of equilibria, which is assumed to contain a saddle. Then Ω_* attracts the orbits, but it is not stable and the global omega limit $\omega_*(Y)$ contains the unstable manifold of the saddle, which is not contained in Ω_*.

Further comments on these issues of reduced omega limits are given in Chapters 6, 9 and 10.

Remark. It is important to notice that all three hypotheses are necessary for Theorem 1.4 to hold. Indeed, it is not difficult to find examples of simple dynamical systems for which only one of the three conditions fails, and then the result is not true. We ask the reader to supply three counter-examples corresponding to each of the hypotheses.

1.6 A result on attractors

This section contains extra material on the attractive properties involved in our main result and is not directly needed in the applications. The S-Theorem can be formulated as the property of a certain set Ω_* to attract individual orbits of $\mathcal{L}$ in the metric of X, i.e., that for every orbit $u(t) \in \mathcal{L}$, we have $d(u(t), \Omega_*) \to 0$ as $t \to \infty$. A natural question in dynamical systems is whether it also attracts compact families of orbits in the same uniform way, by which we mean that given a compact family of data $u(0) \in B$ and given $\varepsilon > 0$, there exists $t_0 > 0$ such that

$$d(u(t), \Omega_*) \leq \varepsilon \qquad \text{for every } u \text{ such that } u(0) \in B, \text{ and every } t \geq t_0.$$

The following result, proved in Hale's book [189], Chapter 3, solves the passage from attraction of orbits (called there points) to the attraction of compact families (compact sets) for continuous semigroups (as those generated by an autonomous equation).

Proposition 1.9 *Let $J \subset X$ be a compact, invariant and stable set for a continuous semigroup $\{S(t), t \geq 0\}$. If J attracts points in a neighbourhood of J, then it attracts compact subsets of a bounded neighbourhood of J.*

We only know how to pass from point attraction to attraction of compact sets in the setting of perturbed dynamical systems by imposing a rather strong hypothesis on the smallness of the action of the perturbation in (1.3) on Ω_* as $t \to \infty$. This is formulated as
(H4) We assume that $\Omega_* \subseteq X$ is closed and stable under asymptotic perturbations in the sense that for every $\varepsilon > 0$, there exist a neighbourhood U of Ω_* and $\delta > 0$ small enough and $t_0 > 0$ large, such that whenever $u \in \mathcal{L}$ and $d(u(t_1), \Omega_*) < \delta$ for $t_1 \geq t_0$, then

$$d(u(t), \Omega_*) < \varepsilon \quad \text{for every } t \geq t_1. \tag{1.23}$$

We state our theorem for sets of orbits of (1.3) in the subclass $\mathcal{L}_0$ defined for $t \geq 0$, and call the set of initial data $X_0 = \{u(0) : u \in \mathcal{L}_0\}$. We denote by $u(t; x)$ the solution with initial data $u(0; x) = x \in X_0$.

Theorem 1.10 *Suppose that we have assumptions* (H1), (H2), (H3) *in the setting of Section* 1.3 *and* (H4), *and assume that the orbits of $\mathcal{L}_0$ depend continuously on their data at $t = 0$. Then Ω_* attracts compact sets of orbits for equation* (1.3) *in the sense that*

$$d(u(t; x), \Omega_*) \to 0 \quad \textit{as } t \to \infty$$

uniformly in $x \in E$, where E is any relatively compact subset of $X_0 \subset X$.

Proof. We consider a compact set $E \subset X_0$. Since Ω_* attracts the orbits of (1.3) by the S-Theorem, for every $x \in E$, the orbit $u(t; x)$ with initial data x converges to Ω_*. Therefore, for every $\varepsilon > 0$, there exists $t(x) > 0$ such that $d(u(t(x); x), \Omega_*) < \delta/2$. By continuity, there exists an open neighbourhood $U(x)$ of x in X such that $d(u(t(x); y), \Omega_*) < \delta$ for any $y \in U(x)$. We may take $t(x) \geq t_0$ of hypothesis (H4). Now, the compactness of E implies that there is a finite covering of E by these neighbourhoods: $E \subset U(x_1) \cup \cdots \cup U(x_n)$. Then for every $t \geq T = \max\{t(x_i) : i = 1, ..., n\}$ and every $x \in E$, we have $x \in U(x_i)$ for some i, hence by (H4)

$$d(u(t; x), \Omega_*) < \varepsilon.$$

This completes the proof. □

Remarks and comments on the literature

1.1. The S-Theorem was introduced by the authors in the paper [169], 1991, in the study of the critical case of the PME with absorption, see Chapter 4. The reduced omega-limit and reduced stability were used for the first time in [173] in the description of regional blow-up for the semilinear heat equation, which is discussed in

Chapter 9. A survey of applications of this stability theorem is presented in [176]. In the applications to PDEs that follow, X is a subspace of an infinite-dimensional functional space, typically $L^p(D)$ for some domain $D \subseteq \mathbb{R}^N$ and some $p \in [1, \infty]$.

In problems discussed in this book, some of them having really nontrivial dynamics, the set Ω_* happens to be one point. It is remarkable that in those cases the S-Theorem provides a *complete identification* of the omega-limits of the perturbed equation without using any specific information about the vanishing perturbation $\mathbf{C} = \mathbf{A} - \mathbf{B}$, apart from hypothesis (H2).

However, there are applications in which the set Ω_* appearing in the theorem is too large, i.e., $\omega(u) \neq \Omega_*$, even in the reduced case, and then further independent analysis is needed to determine the range of $\omega(u)$ inside Ω_*. The *selection rule* may take different forms depending on the problem under consideration. This is where more precise properties of the perturbation $\mathbf{C}$, or the estimate of its size, enter the picture. Thus, in Chapter 4 we perform the selection by using the fact that $\mathbf{C}(u, t)$ is not integrable in time for some fixed $u = f \in \Omega_*$. More specifically, in that problem Ω_* consists of fixed points for equation (1.4) and we exclude from $\omega(u)$ all points (i.e., functions) of Ω_* but one, because the perturbation is not integrable, $\mathbf{C}(f, t) \notin L^1(\mathbb{R}_+ : X)$ for those $f \in \Omega_*$. Note that this lack of integrability is compatible with the assumption that $\mathbf{C}$ is asymptotically vanishing. A different selection rule is based on the transversality and intersection properties of the orbits of family $\mathcal{L}$ with respect to a subset of orbits of $\mathcal{L}_*$ in a neighbourhood of Ω_*. Such a technique is introduced in Chapters 5, 9, 10 and 12, and the properties are shown to hold because the families are solutions of one-dimensional second-order parabolic equations to which the Sturmian theory of *Intersection Comparison* applies.

1.2. There is a huge literature on asymptotic methods for dynamical systems, especially after the work of Lasalle [236] for autonomous ordinary differential equations. We refer the reader to the works by Babin and Vishik [23], Dafermos [89], Hale [189], Ladyzhenskaya [232], Henry [191], Temam [297] for PDEs and infinite-dimensional systems. An abstract approach is performed in [50], [289].

1.3. The concept of limiting equations is a basic concept in the theory of singular perturbations, for instance in Prandtl's boundary layer theory [288]. Most of the work on perturbation concerns problems which have a small parameter ε, and the perturbation does not vanish with time as in the present case. Limit equations for ODEs with asymptotically vanishing perturbations have been considered by several authors, like Markus [245] and Artstein, see Appendix A in [236].

2

Nonlinear Heat Equations: Basic Models and Mathematical Techniques

This chapter collects a series of results on the theory of nonlinear heat equations that may be useful to the reader for the correct understanding of subsequent chapters. Some of the material informs about the general philosophy of these equations, a second part develops standard subjects of this theory that allow the reader to compare the standard methods with the ones based on the S-Theorem. In particular, we use the example of the porous medium equation as a convenient setting to introduce the techniques of fixed and continuous scaling that play a big role in the theory of asymptotic analysis of nonlinear problems. Scaling is our way of presenting what is also known in the literature as renormalization. *The last part contains technical results that will be used later on.*

2.1 Nonlinear heat equations

We want to remind the reader of some facts that underline our work in subsequent chapters. Different linear, semilinear and quasilinear versions of parabolic second-order equations enter many of the textbooks on the theory of PDEs and several fields of their applications. By a nonlinear heat equation we mean a second-order evolution PDE formally of parabolic type, loosely speaking a variation of the classical heat equation. Apart from their interest as mathematical models in the applied sciences, this class of equations is mathematically interesting because in a certain sense it exhibits minimal complexity. This is an interesting dynamical feature, it suggests to the researcher the idea that such equations should be understood first.

In our examples we usually deal with two differential operators acting simultaneously: a diffusion (second-order) operator plus another, typically lower-order operator describing, e.g., a process of reaction, convection or absorption. We will be specially interested in cases in which both processes act opposite to each other and have similar strength. The nonlinear interaction between such operators makes the mathematics of the equations nontrivial. Indeed, we usually work with two operators (sometimes, a single one to be split by means of a nonlinear transformation) of not more than the second order of differentiation. On the other hand, this setting describes various nonlinear phenomena, arising in quite different fields of application, like nonlinear heat conductivity, combustion, detonation, filtration of gases and liquids in porous media and plasma physics. Indeed there have been a lot of beautiful general results obtained in such problems in the last fifty years, especially those which deal with fundamental mathematical questions of existence, uniqueness, regularity of weak (generalized) solutions, as well as optimal conditions on their global

V. A. Galaktionov et al., *A Stability Technique for Evolution Partial Differential Equations*

solvability. We are not going to treat most of them in any detail, and actually it is impossible in a single book. We consider nonlinear equations for which local existence, uniqueness and general regularity are well known nowadays.

Semilinear, quasilinear and nonlinear heat equations

Let us describe next the main types of nonlinear heat equations that will appear in the sequel. The main diffusion operators to be considered are those from the *heat equation*

$$u_t = \Delta u \quad \text{in } Q = \mathbb{R}^N \times \mathbb{R}_+, \tag{2.1}$$

where Δ is the spatial Laplacian in $\mathbb{R}^N$, $\Delta = \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2}$, and its variants where Δu is replaced by an elliptic operator of the form

$$Au = \sum_{(i,j)} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} \quad \text{or} \quad Au = \sum_{(i,j)} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial u}{\partial x_i}\right).$$

The multiple applications of the heat equation are well known in the mathematical, physical or engineering literature.

We then have the nonlinear models like the *porous medium equation* (PME)

$$u_t = \Delta u^m, \quad m > 1 \quad (u \geq 0), \tag{2.2}$$

which is called the *fast diffusion equation* if $m \in (0, 1)$. This equation occurs in diffusion of liquids and gases in porous media and in processes of electron and ion conductivity in plasma, and in all these applications the restriction $u \geq 0$ applies. But it is mathematically interesting to allow for negative values of the unknown $u = u(x, t)$. In that case, we need to redefine the equation in a suitable way for it to be still parabolic. Our choice is

$$u_t = \Delta(|u|^{m-1}u) = \nabla \cdot (m|u|^{m-1}\nabla u). \tag{2.3}$$

The investigation of the existence of generalized solutions of the initial and boundary-value problems for the equations of nonlinear diffusion of those types has been extended to more general forms, like the so-called *filtration equation*

$$u_t = \Delta\Phi(u) + f, \tag{2.4}$$

where Φ is a monotone nondecreasing function and $f \in L^1_{loc}(Q)$. An interesting equation of this type with exceptional nonpower nonlinearities is the *equation of superslow diffusion*

$$u_t = (e^{-1/u})_{xx}, \tag{2.5}$$

where the nonlinearity $\Phi(u) = e^{-1/u}$ decays as $u \to 0$ faster than any power law $\Phi(u) = u^m$ in the PME. In a formal asymptotic sense, for a certain class of solutions, this equation corresponds to $m = \infty$, a *critical* exponent.

Another fruitful direction of extension is given by the *p-Laplacian equation* (PLE)

$$u_t = \Delta_p u \equiv \nabla \cdot (|\nabla u|^{p-2} \nabla u), \quad p > 1 \quad (\nabla = \text{grad}_x) \tag{2.6}$$

and its many variants, all of them *quasilinear*, (formally) parabolic equations. The p-Laplacian operators are typical for nonNewtonian (dilatant) liquids.

Adding lower-order terms to the right-hand side is an important option, and the zero-order term represents an absorption or a reaction, depending on its sign. In the latter case, when this term is positive or at least nonnegative (a source-like term), we obtain, for instance, the *semilinear reaction heat equation*

$$u_t = \Delta u + f(u), \tag{2.7}$$

where the exponential source term $f(u) = e^u$ corresponds to the famous nonstationary *Frank–Kamenetskii equation* in combustion theory formulated in 1938. The power approximation $f(u) = u^p$ leads to the semilinear heat equation

$$u_t = \Delta u + u^p, \quad p > 1. \tag{2.8}$$

Such an approximation makes sense for the PME operator, which gives the quasilinear heat equation

$$u_t = \Delta u^m + u^p, \quad m > 1, \; p > 1, \tag{2.9}$$

and for the p-Laplacian version $u_t = \Delta_p u + u^q, \quad p > 2, \; q > 1$. More combinations are possible and appear both in the mathematical and the applied literature.

In the above equations, the nonlinear interaction of the operators creates interesting structures. We will focus on the generic ones, which exhibit a stable spatio-temporal structure. Sometimes, we are interested in the corresponding countable spectrum of patterns.

It is most interesting that when these equations contain strong superlinear combustion terms they exhibit an important nonlinear phenomenon called *blow-up* when the solutions become infinite in finite time, i.e., as $t \to T < \infty$. This corresponds to the effect of adiabatic explosion in combustion theory. Such a highly nonstationary behaviour of the reaction-diffusion process creates a series of interesting mathematical problems, in particular, the asymptotic behaviour of solutions as $t \to T$. This is an important subject of study of the present book. We will consider one case of blow-up solutions to a semilinear equation with a *nonlocal* nonlinearity.

On the other hand, if we add a negative term to the diffusion-like operators, we obtain quasilinear heat equations with *absorption*. For instance, the *PME with absorption*

$$u_t = \Delta u^m - u^p. \tag{2.10}$$

Though the absorption term prohibits the growth of solutions and blow-up is impossible, we arrive at the questions of asymptotic behaviour, as $t \to \infty$, of bounded solutions, where we want to know the rate of decay and shape of the asymptotic profile.

We show that in different parameter ranges such a behaviour can be quite different, especially for some special critical exponents. In the case of strong, $p \in [0, 1)$, or singular, $p < 0$, absorption, bounded solutions extinguish in finite time, as $t \to T$, thus creating an interesting finite-time extinction asymptotic behaviour.

We introduce and study some other singular phenomena for quasilinear parabolic equations including fully nonlinear equations, of the general form

$$u_t = F(u, Du, D^2u) \tag{2.11}$$

with suitable F monotone in the last argument. A simple example of such equations is the so-called *dual PME*,

$$u_t = |\Delta u|^{m-1} \Delta u, \quad m > 1,$$

which appears in elastoplastic media. Finally, the transformations and limit processes that we perform on our equations will lead us into so-called Hamilton–Jacobi equations,

$$u_t = F(Du),$$

or their viscous counterpart, $u_t = \varepsilon \Delta u + F(Du)$, along with many variants, see Chapters 5, 10. The most typical Hamilton–Jacobi equation is the *Eikonal Equation*, $u_t = |\nabla u|^2$.

We will give further details as the equations appear in the corresponding chapters.

2.2 Basic mathematical properties

In most of the cases, we consider nonnegative solutions $u = u(x, t) \geq 0$. The nonnegativity property of the solutions is guaranteed by the maximum principle which applies to all the equations.

The heat equation

In the case of the heat equation and its uniformly parabolic linear variants, the results are well known. Thus, the Cauchy problem for the heat equation with initial data

$$u(x, 0) = u_0(x), \quad x \in \mathbb{R}^N, \tag{2.12}$$

admits a unique solution $u(x, t)$ defined in $Q = \mathbb{R}^N \times \mathbb{R}_+$ under the following conditions: u_0 is locally integrable and

$$\int_{\mathbb{R}^N} |u_0(x)|\, e^{a|x|^2}\, dx < \infty \tag{2.13}$$

for any $a > 0$. If we impose the restriction that this is true for any $a < a_0$, then the solution exists in some time interval $0 < t < T$. The solution can be represented by the convolution with the *fundamental solution* of operator $L = \partial/\partial t - \Delta$,

$$u(x,t)=\frac{1}{(4\pi t)^{N/2}}\int_{\mathbb{R}^N} u_0(y)\exp\{-|x-y|^2/4t\}\,dy. \tag{2.14}$$

See Figure 2.1.

It follows that the solution depends continuously on the data in various norms, so that the problem is well posed. It is often convenient to make a particular choice of such possible spaces. The most typical for our purposes is $X = L^1(\mathbb{R}^N)$, and then we look at the collection of maps $S_t : u_0 \mapsto u(t)$, where we write $u(t) = u(\cdot, t) \in X$. The existence and uniqueness result can be reformulated as the existence of a *continuous semigroup* in X. In fact, it is a contraction semigroup:

Theorem 2.1 *The maps $S_t : u_0 \mapsto u(t)$ are order-preserving contractions on $X = L^1(\mathbb{R}^N)$. More precisely, for $t > 0$,*

$$\|(u_1(t)-u_2(t))_+\|_{L^1(\mathbb{R}^N)} \le \|(u_1(0)-u_2(0))_+\|_{L^1(\mathbb{R}^N)}, \tag{2.15}$$

where $(\cdot)_+$ denotes the positive part, $\max\{\cdot, 0\}$. *In particular, plain L^1-contraction holds:*

$$\|u_1(t)-u_2(t)\|_{L^1(\mathbb{R}^N)} \le \|u_1(0)-u_2(0)\|_{L^1(\mathbb{R}^N)}. \tag{2.16}$$

The maximum principle also follows from property (2.15).

Property (2.15) is called T-contraction and was introduced by Bénilan in order to tie together the concepts of contraction and order. We will see that this property extends to other initial- and boundary-value problems for many of the nonlinear heat equations we consider. Actually, the property of T-contraction holds for the heat equation not only when $X = L^1(\mathbb{R}^N)$, but also in the Lebesgue spaces $X = L^p(\mathbb{R}^N)$ with any $1 \le p \le \infty$.

Supersolutions and subsolutions. The maximum principle is applied in comparison theorems: in order to show that two solutions are ordered, e.g., $u_1 - u_2 \ge 0$, we apply the maximum principle to the difference in a suitable parabolic domain, not necessarily a strip of the form $\mathbb{R}^N \times (0, T)$, and check that $u_1 - u_2 \ge 0$ on the *parabolic boundary*. Let us point out a crucial detail for the applications: the conclusion holds even when u_1, u_2 are not solutions, if u_1 is a *supersolution* and u_2 a *subsolution*. In the first case, we ask that $u_{1,t} - \Delta u_1 \ge 0$, in the second, $u_{2,t} - \Delta u_2 \le 0$. Super- and subsolutions (also known as upper and lower solutions) satisfying the corresponding parabolic differential inequalities are well known in the classical parabolic and elliptic theory, hence, we will not insist at this point. Note only that by approximation these ideas extend to the usual classes of nonlinear degenerate equations of elliptic and parabolic types, though they do not extend to higher-order equations.

Strong maximum principle. Besides, the maximum principle admits a strong form that can be formulated as follows: if the initial data $u_0 \ge 0$ a.e. in $\mathbb{R}^N$, $u_0 \not\equiv 0$, then the solution is positive everywhere for $t > 0$, $u(x,t) > 0$ in Q. This property fails for many of the nonlinear diffusion models in various degrees, and such failure is tied to difficulties in the regularity theory.

Another important aspect of the heat equation theory is regularity: the solutions given by formula (2.14) are C^∞ smooth in Q even for general data as stated above (regularizing effect). This is true for all equations of the form $u_t = Au + f(u)$, where A is a uniformly elliptic operator with constant coefficients and f is, say, a C^∞ real function with linear growth as $u \to \infty$. But the property will dramatically fail for the nonlinear models.

Other problems. The heat equation is also typically posed in bounded space domains with Dirichlet, Neumann or other boundary conditions. Nonlinear theory appears when we put nonlinear boundary conditions of the form

$$\frac{\partial u}{\partial n} + \beta(u) = g(x),$$

where β is a real function. But a main area of development of nonlinear analysis is that of *free boundary problems* for the heat equation, where the space domain is allowed to change to respond to overdetermined boundary conditions. The most famous of these problems is the *Stefan Problem*, the typical prototype for the description of the phenomena of change of phase. Much attention has been also given to the free boundary problem with fixed-gradient conditions of combustion theory, where we impose the conditions $u = 0$, $|\nabla u| = c > 0$ on the moving boundary.

The porous medium equation

We will also focus on the Cauchy problem, but we will restrict most of our attention to the case of nonnegative solutions, $u \geq 0$. The Cauchy problem (2.2), (2.12) does not possess classical solutions for general data in the class $u_0 \in L^1(\mathbb{R}^N)$, $u_0 \geq 0$ (or even in a smaller class, like the set of smooth nonnegative and rapidly decaying initial data). This is due to the fact that the equation is parabolic only where $u > 0$, but degenerates at the level $u = 0$. This has, as a consequence, finite propagation, whereby, for instance, a solution with compactly supported initial data preserves the property for all later times. A free boundary or interface appears to separate the sets $\{u = 0\}$ and $\{u > 0\}$. It can be proved that near moving interfaces solutions cannot be very smooth. We need to introduce a concept of generalized solution and make sure that the problem is well posed in that class. On the other hand, when $u_0 \geq \varepsilon > 0$ the equation does not degenerate, and the quasilinear parabolic theory can be used to produce classical positive solutions $u \geq \varepsilon$. This is a considerable help in building the generalized theory. These are the main features.

Definitions. By a *solution of equation* (2.2) we will mean a nonnegative function $u(x, t)$ defined for $(x, t) \in Q$ such that
(i) viewed as a map

$$t \to u(\cdot, t) = u(t), \tag{2.17}$$

we have $u \in C\big((0, \infty) : L^1(\mathbb{R}^N)\big)$,
(ii) the functions u^m, u_t and Δu^m belong to $L^1((t_1, t_2) : L^1(\mathbb{R}^N))$ for all $0 < t_1 < t_2$,
(iii) equation (2.2) is satisfied in the sense of distributions in Q.

By a *solution of the Cauchy problem* we mean a solution of (2.2) such that the initial data are taken in the following sense:

$$u(t) \to u_0 \quad \text{in} \quad L^1(\mathbb{R}^N) \quad \text{as } t \to 0. \tag{2.18}$$

In other words, $u \in C([0,\infty) : L^1(\mathbb{R}^N))$ and $u(0) = u_0$.

This definition is usually called in the specialized literature a *strong solution*. It is suitable for our purposes since the Cauchy problem is well posed in this setting, but it is not the unique choice; we could have used the concept of *weak solution*, where we merely ask u^m and $\nabla_x u^m$ to be locally integrable functions in $\mathbb{R}^N \times [0,\infty)$ and the equation is satisfied in the sense that

$$\iint \{u\varphi_t - \nabla_x u^m \cdot \nabla_x \varphi\}\, dx\, dt + \int u_0(x)\varphi(x,0)\, dx = 0 \tag{2.19}$$

holds for every smooth test function $\varphi \geq 0$ which vanishes for all large enough $|x|$ and t. See [304] for a discussion of those equivalent alternatives.

Theorem 2.2 *The Cauchy problem is well posed in the framework of strong solutions. The maps $S_t : u_0 \mapsto u(t)$ form a continuous semigroup in $L^1(\mathbb{R}^N)$.*

The orbits $t \mapsto u(t)$ of the continuous semigroup are continuous functions from $[0,\infty)$ into $L^1(\mathbb{R}^N)$. Here are some of the main properties of the solutions.

Property 1. CONTRACTION. *For every two solutions u_1 and u_2, we have*

$$\|(u_1(t) - u_2(t))_+\|_{L^1(\mathbb{R}^N)} \leq \|(u_1(0) - u_2(0))_+\|_{L^1(\mathbb{R}^N)}, \quad t > 0. \tag{2.20}$$

This is the T-contraction property.

When we apply the principle to the pairs (u_1, u_2) and (u_2, u_1) and sum the results we get plain L^1-contraction

$$\|u_1(t) - u_2(t)\|_{L^1(\mathbb{R}^N)} \leq \|u_1(0) - u_2(0)\|_{L^1(\mathbb{R}^N)}. \tag{2.21}$$

In other words, the maps $S_t : u_0 \mapsto u(t)$ form an order-preserving semigroup of contractions on $L^1(\mathbb{R}^N)$. Let us point out a main difference with the heat equation: contraction is not proved in any space $L^p(\mathbb{R}^N)$ with $1 < p \leq \infty$.

Property 2. MAXIMUM PRINCIPLE. The maximum principle follows from this property when we apply it to solutions such that $u_1(0) \leq u_2(0)$, i.e., when $(u_1(0) - u_2(0))_+ = 0$, since this value is preserved for $t > 0$. Hence, $u_1(0) \leq u_2(0)$ a.e. implies $u_1(t) \leq u_2(t)$ a.e. for all $t > 0$.

On the other hand, the strong maximum principle does not hold for solutions which touch the degenerate level, $u = 0$: the existence of free boundaries, to be discussed below, is a witness to that fact.

Property 3. MASS CONSERVATION. *The solutions of the Cauchy problem satisfy the law of mass conservation*

$$\int_{\mathbb{R}^N} u(x,t)\,dx = \int_{\mathbb{R}^N} u_0(x)\,dx. \tag{2.22}$$

When $u \geq 0$ this means $\|u(t)\|_1 = \|u_0\|_1$ for all $t > 0$. The property will also be true for solutions of any sign, but then it does not imply conservation of L^1-norm. It is also true for $0 < m < 1$ if $m \geq (N-2)/N$, but not below that value.

Property 4. SOURCE-TYPE SOLUTIONS. The PME is a nonlinear equation and there is no equivalent to the representation formula (2.14). However, there is a particular family of solutions that plays a role equivalent in some sense to the fundamental solution for the heat equation. Indeed, the PME with $m > 1$ admits a one-parameter family of special solutions

$$\mathcal{U}(x,t;C) = t^{-\alpha} F(xt^{-\beta};C), \tag{2.23}$$

with parameter $C > 0$. The functions $\mathcal{U}(x,t;C)$ were variously called *source-type solutions, fundamental solutions, Barenblatt–Pattle solutions, Zel'dovich–Kompaneetz–Barenblatt solutions* (ZKB), the last being our preferred option in this text. They are given by the explicit formula

$$F(\eta) = (C - k\,|\eta|^2)_+^{\frac{1}{m-1}}, \quad \alpha = \frac{N}{N(m-1)+2}, \; \beta = \frac{1}{N(m-1)+2}. \tag{2.24}$$

F is called the profile, α and β are the similarity exponents. $C > 0$ is a free constant and k is fixed, $k = (m-1)\beta/2m$. We have $\mathcal{U}^{m-1} = (C\,t^{2\beta} - k\,|x|^2)_+/t$. See Figure 2.2. The fact that the fundamental solutions are self-similar is important in what follows, the fact that they are explicit is not.

Comparison of infinite space propagation for $m = 1$ and finite propagation for $m > 1$.

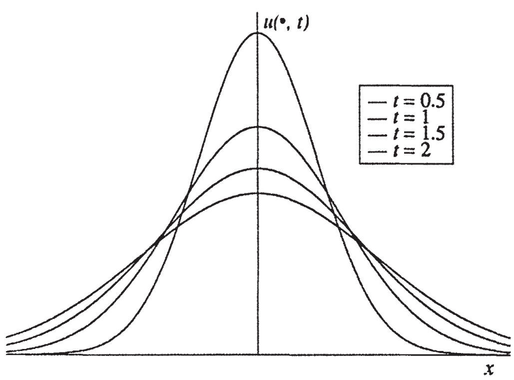

Fig. 2.1. Heat Equation. Profile of the fundamental solutions of the HE at different times.

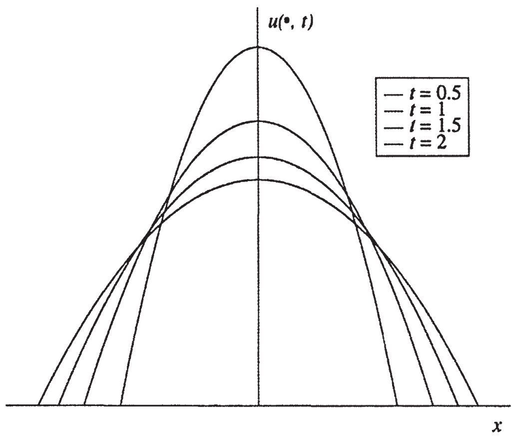

Fig. 2.2. Porous Medium Equation. Profile of the ZKB solutions at different times.

The investigation of the PME during the last decades has shown that the source-type solutions play a big role as a paradigm of the properties and behaviour of wide classes of solutions in several respects, most notably, in what concerns regularity and large-time behaviour.

We point out that the $\mathcal{U}(x, t; C)$ are strong solutions of (2.2), but, strictly speaking, they are *not* solutions of the Cauchy problem because they do not take L^1 initial data. Indeed, it is easy to check that $\mathcal{U}$ converges to a Dirac mass as $t \to 0$ (this is the reason for the name "source-type solutions"). This fact acts strongly in the proof of asymptotic convergence, cf. Subsection 2.3.

Property 5. BOUNDEDNESS. *Solutions with L^1 data are bounded for $t \geq \tau > 0$. Moreover, there exists a constant $C = C(m, N) > 0$ such that*

$$0 \leq u(x,t) \leq C\, \|u_0\|_1^{2\beta}\, t^{-N\beta} \quad \textit{with } \beta = 1/[N(m-1)+2]. \tag{2.25}$$

This is the so-called $L^1 \to L^\infty$ effect. The maximum principle implies on the other hand an $L^\infty \to L^\infty$ effect: $\|u(t)\|_\infty \leq \|u(0)\|_\infty$. This can be easily extended to an $L^p \to L^\infty$ effect for any $1 \leq p < \infty$ (by interpolation).

Property 6. LIMITED REGULARITY. *Bounded solutions are uniformly Hölder continuous for $t \geq \tau > 0$. This statement cannot be improved for general solutions.*

Therefore, strong solutions are continuous but have limited additional regularity. The source-type solutions are an example of limited regularity. A finer question is optimal regularity, i.e., finding the best Hölder exponent. In one dimension the answer is $\alpha = \min\{1, 1/(m-1)\}$, which corresponds to the property of Lipschitz continuity of the pressure $v = u^{m-1}$. The question is not completely settled for $N > 1$, where the exponent is lower, since v need not be Lipschitz continuous. Here is a situation where the source-type solutions are not the paradigm.

Property 7. APPROXIMATION. We recall that all solutions with positive data are positive everywhere and C^∞ smooth. Combined with the L^1 contraction, this implies that every strong solution is the limit of smooth and positive solutions, with approximation in the norm of $L^\infty(\mathbb{R}_+ : L^1(\mathbb{R}^N))$. The local regularity implies that the convergence also takes place locally uniformly in Q.

Property 8. FINITE PROPAGATION PROPERTY. If the initial function u_0 is compactly supported, so are the functions $u(\cdot, t)$ for every $t > 0$. Under these conditions there exists a free boundary or interface $\Gamma(u)$ that separates the regions $\mathcal{P}(u) = \{(x,t) \in Q : u(x,t) > 0\}$ and $\{(x,t) \in Q : u(x,t) = 0\}$. It is precisely defined as the boundary of the positivity set

$$\Gamma(u) = \partial\mathcal{P}(u). \tag{2.26}$$

Equivalently, it can also be defined as the boundary of the support of u, which is the closure of $\mathcal{P}(u)$. According to [68] this interface is an N-dimensional Hölder continuous hypersurface in $\mathbb{R}^{N+1}$.

We do not need to start with a compactly supported solution to have a free boundary $\Gamma \neq \emptyset$, since the property of finite propagation is quite general. Γ is nonempty as long as u_0 vanishes on a set that contains a ball. Even vanishing at one point x_0 will do, depending on the behaviour of u_0 near x_0. In this extreme case Γ is an interval of the form $\{(x_0, t) : 0 \le t \le t_w\}$ for some $t_w \ge 0$ called the waiting time.

On the other hand, finite propagation has the further property that the support is noncontracting in time (this is sometimes called *retention property*, because points of positivity are conserved). Moreover, the support eventually reaches all points of the space (*penetration property*).

Property 9. ENERGY ESTIMATES. Another aspect of the regularization of solutions with time is obtained by multiplying the equation by u^m and formally integrating by parts. We arrive at

$$\int_{\mathbb{R}^N} u^{m+1}(x,t)\,dx + (m+1)\int_\tau^t\!\!\int_{\mathbb{R}^N} |\nabla(u^m)|^2\,dx\,dt \le \int_{\mathbb{R}^N} u^{m+1}(x,\tau)\,dx \tag{2.27}$$

for all $0 < \tau < t$. Since we know by the previous properties that $u(\tau) \in L^p(\mathbb{R}^N)$ for all $p > 1$, in particular $p = m+1$, we conclude that ∇u^m is uniformly bounded in $L^2(\mathbb{R}^N \times (\tau, t))$ in terms of the mass of the initial data. The justification of the calculation can be found in [304]. In the same spirit, multiplication by $(u^m)_t$ and integration by parts gives

$$c\int_\tau^t\!\!\int_{\mathbb{R}^N} \big((u^{\frac{m+1}{2}})_t(x,t)\big)^2 dx\,dt + \int_{\mathbb{R}^N} |\nabla u^m(x,t)|^2 dx \le \int_{\mathbb{R}^N} |\nabla u^m(x,\tau)|^2 dx, \tag{2.28}$$

where $c = 8m/(m+1)^2$. Combining with the previous one, it gives a bound for $\partial(u^{(m+1)/2})/\partial t$ in $L^2(\mathbb{R}^N \times (\tau, t))$ in terms of the mass of the initial data, and a better bound for ∇u^m in $L^\infty((\tau, \infty) : L^2(\mathbb{R}^N))$. These and other gradient estimates were developed by Bénilan, cf. [36].

The next estimate is due to Aronson–Bénilan [16] and plays a big role in the study of the Cauchy problem for the PME.

Property 10. FUNDAMENTAL REGULARITY ESTIMATE AND CONSEQUENCES. *Any nonnegative solution of the Cauchy problem satisfies the estimate*

$$\Delta(u^{m-1}) \ge -\frac{C}{t}, \quad \text{where } C = \frac{\alpha\,(m-1)}{m}, \quad \alpha = \frac{N}{N(m-1)+2}. \tag{2.29}$$

This implies another interesting estimate: $u_t \ge -\alpha\, u/t$. *Moreover, conservation of mass is equivalent to* $\int u_t\,dx = 0$, *so that the last estimate leads to*

$$\int |u_t(x,t)|\,dx \le \frac{2\alpha}{t}\int u(x,t)\,dx\,. \tag{2.30}$$

The latter is an special estimate because it is not one-sided. On the other hand, the one-sided estimate (2.29) is exact precisely for the source-type solutions (2.23) that play a key role in our theory.

Property 11. SCALING. One of the most important properties of the PME is *scaling invariance.* It is simpler to state in terms of the pressure $v(x,t) = u^{m-1}$. The assertion says that any pressure solution $v(x,t)$ will produce a family of pressure solutions by means of the formula

$$\tilde{v}(x,t) = \frac{B}{A^2} v(Ax, Bt) \tag{2.31}$$

for any $A, B > 0$. The choice $B = A^{2+N(m-1)}$ is precisely the scaling that conserves mass for the density u.

Property 12. OTHER CLASSES OF DATA. The theory need not be confined to data in L^p spaces. Optimal conditions on the initial data that produce a weak solution defined in a domain $Q_T = \mathbb{R}^N \times (0,T)$ are known and take the form

$$\limsup_{R\to\infty} R^{-[N+2/(m-1)]} \int_{|x|\le R} u_0(x)\,dx < \infty.$$

These conditions generalize the condition of square exponential growth that is well known for the heat equation, and allow for the Cauchy problem to be well posed for nonnegative solutions in a class of optimal initial data.

On the other hand, the existence and uniqueness theory extends to data and solutions of any sign when the equation is written in the form $u_t = \Delta(|u|^{m-1}u)$, in the standard setting $u_0 \in L^1(\mathbb{R}^N)$, where it still generates a semigroup of contractions in $L^1(\mathbb{R}^N)$, or in classes of growing data. It must be remarked that the mathematical theory is less polished, and the interest for the applications is up to now smaller.

Super- and subsolutions. Generalizing what was said for classical solutions of the heat equation, it is natural in the PME to define the classes of weak super- and subsolutions by slightly changing the definition. Thus, for a supersolution equation (2.19) becomes

$$\iint \{u\varphi_t - \nabla_x u^m \cdot \nabla_x \varphi\}\,dx\,dt + \int u_0(x)\varphi(x,0)\,dx \le 0 \tag{2.32}$$

with the same test functions: $\varphi \ge 0$ is smooth and vanishes for all large enough $|x|$ and t. The sign is reversed for a subsolution. Then, the T-contraction is still valid for $u_1 - u_2$ since u_1 is a sub- and u_2 is a supersolution. The comparison on parabolic domains is still valid.

Fast diffusion. Let us note that part of this scenario is still true for the range $m < 1$, called *fast diffusion* (the diffusivity coefficient is unbounded at $u = 0$). Actually, most of the properties hold as long as $m > (N-2)_+/N$. The main difference is that finite propagation does not hold: nonnegative solutions of the Cauchy problem become immediately positive everywhere for all $t > 0$.

Other problems. There are a number of initial and boundary value problems that have been studied in connection with the PME. The properties that have been found bear a great resemblance with the list we have exhibited for the Cauchy problem, though the details differ.

The most frequent problem is the Cauchy–Dirichlet problem posed in a bounded space domain $\Omega \subset \mathbb{R}^N$ with homogeneous boundary conditions, $u(x,t) = 0$ on $\partial\Omega \times (0,\infty)$. Again, we get limited regularity of nonnegative solutions, finite propagation, it generates a T-contraction semigroup in $L^1(\mathbb{R}^N)$, and so on. As a difference, the fundamental estimate is replaced by the coarser one

$$u_t \geq -\frac{u}{(m-1)t}.$$

When the boundary data are not zero, some of the estimates are lost, but weaker forms can be chosen. Similar observations apply to the Cauchy–Neumann problem. Other problems that have been investigated in detail are the initial and boundary value problem in a half line $\Omega = (0,\infty) \subset \mathbb{R}$ with either Dirichlet or Neumann boundary data, and also some initial-and-boundary-value problems in exterior domains.

Other equations

In the case of the p-Laplacian equation, $u_t = \Delta_p u$, $p > 2$, the similarity with the properties of the PME is striking: (i) the problem is not well posed in the framework of classical solutions, so that a weak or a strong theory must be introduced; (ii) there is a T-contraction property, but now it works in all L^p spaces, $1 \leq p \leq \infty$; (iii) the maximum principle holds, but not in its strong form; (iv) conservation of mass holds; (v) the source-type solutions also exist and are given by explicit formula

$$\mathcal{U}(x,t;C) = t^{-\alpha} F(xt^{-\beta};C), \tag{2.33}$$

with free parameter $C > 0$, and

$$F(\eta) = (C - k\,|\eta|^{p/(p-1)})_+^{\frac{p-1}{p-2}}, \quad \alpha = \frac{N}{N(p-2)+p}, \quad \beta = \frac{1}{N(p-2)+p}, \tag{2.34}$$

where $k = k(p,N) > 0$. This formula holds even for $p < 2$ as long as $N(p-2)+p > 0$, i.e., $p > 2N/(N+1)$; (vi) boundedness and limited regularity hold, but this time the Hölder space is $C^{1,\alpha}$ in space; (vii) the finite propagation property holds for $p > 2$; (viii) there are energy estimates; (ix) there is a fundamental estimate, similar to Aronson–Bénilan's for the PME,

$$\Delta_p(u^m) \geq -\frac{C}{t}, \tag{2.35}$$

where m is a precise power, $m = (p-2)/(p-1) < 1$, and C is a certain universal constant $C = C(p,N)$, cf. [108]; (x) the scaling rule is

$$\widetilde{u}(x,t) = \left(\frac{B}{A^p}\right)^{1/(p-2)} u(Ax, Bt). \tag{2.36}$$

The situation for existence, uniqueness, estimates and regularity of the other nonlinear models mentioned in the previous section is somewhat intermediate to the models just discussed.

2.3 Asymptotics

The mathematical theory of nonlinear heat equations includes as an important subject the study of the asymptotic behaviour of solutions, which is a fundamental question for the applications. The asymptotic problem corresponds to long time behaviour if the solutions are global in time, to finite-time if they have blow-up.

Though the PDEs we consider can be viewed as *infinite-dimensional dynamical systems*, their strong nonlinear dissipativity properties often play a constructive role in establishing a lower (or even finite) dimension for the corresponding asymptotic attractors. The structure of such attractors and the omega-limits of each individual orbit (from a suitable class of solutions) are the main questions of the general asymptotic theory of nonlinear PDEs. In the theory of finite-dimensional dynamical systems, a general result for hyperbolic equilibria is known, the *Hartman–Grobman theorem*. No result of such kind and generality is available for nonlinear PDEs.

It turned out in the last thirty years of very extensive development of asymptotic methods that nonlinear heat equations, even of a simple form with quadratic or power-like nonlinearities, can exhibit sophisticated and unusual asymptotic properties. A common feature of such complex asymptotic problems is that in the natural rescaled sense, the global asymptotic structure of the orbits is driven by a *nonautonomous* infinite-dimensional dynamical system. This creates an interesting object for the general asymptotic theory. Each asymptotic problem from such a class turns out to be very individual, classical asymptotic methods often fail and the asymptotic analysis needs special new techniques. A classification of asymptotic patterns becomes a complicated problem. In fact, it is not exaggerated to say that, unlike the local existence, uniqueness and regularity problems which have been treated by unified approaches, the hardest asymptotic problems for nonlinear heat equations remained open for a long period. A lot of complicated asymptotics for nonlinear heat equations were discovered in the theory of *blow-up* in nonlinear diffusive media with combustion terms, that have important applications. Finite-time blow-up often exhibits unusual behaviour, and the asymptotic analysis always reduces to the study of the evolution orbits on essentially unstable manifolds. The structurally stable behaviour is then obtained via special rescaling of the orbits.

Turning to more concrete questions, the asymptotic behaviour of the solutions of the equations under consideration depends on the type of initial and boundary conditions. It is therefore a quite large subject, and this leads us to make the following restrictions.

(1) We are mainly interested in the presentation of the results for the PME as a key example of nonlinear behaviour, with partial attention to the heat equation and the p-Laplacian equation.

(2) We typically consider nonnegative solutions $u = u(x,t) \geq 0$. As we know, the nonnegativity property of the solutions is guaranteed by the maximum principle which applies to all the equations.

(3) We consider either the Cauchy problem posed in the whole space with integrable data, or the Dirichlet problem posed in a bounded domain.

On a general level, it has been pointed out in many papers and corroborated by numerical experiments that similarity solutions furnish the asymptotic representation for solutions of a wide range of problems in mathematical physics. The reader is referred to the book of G.I. Barenblatt [27] for a detailed discussion of this subject. Self-similar solutions and the forthcoming scaling techniques will play a prominent role in our asymptotic study.

The reader should note that we are going to describe the behaviour of a whole class of solutions of equations like (2.1), (2.2) or (2.6) in terms of a simple family of functions which are solutions of the equation under consideration; moreover, the models are not in the same class, but in a larger class. More precisely, the special solutions which represent the whole dynamics at the asymptotic level exhibit a *singularity* (at $x = 0, t = 0$). The use of singular solutions is a curious and quite general feature in the theory of asymptotic analysis.

Asymptotics for the heat equation

The asymptotic behaviour of the typical initial and boundary value problems in usual classes of solutions is a well researched subject for the linear heat equation, $m = 1$. The classical result for the Cauchy problem says that under the assumptions of nonnegative and integrable initial data $u_0 \in L^1(\mathbb{R}^N)$, $u_0 \geq 0$, there is convergence of the solution of the Cauchy problem towards the Gaussian kernel

$$u(x,t) \sim \frac{M}{(4\pi t)^{N/2}} \exp\{-|x|^2/4t\}, \tag{2.37}$$

where $M = \int u_0(x)\,dx$ is the mass of the solution (space integration is performed by default in $\mathbb{R}^N$).

In the case of the Cauchy–Dirichlet problem posed in a bounded domain $\Omega \subset \mathbb{R}^N$, it is well known that the asymptotic shape of any solution with nonnegative initial data in $L^2(\Omega)$ approaches one of the special *separate variables solutions*

$$u_1(x,t) = c\,T_1(t)\,F_1(x)\,. \tag{2.38}$$

Here $T_1(t) = e^{-\lambda_1 t}$, where $\lambda_1 = \lambda_1(\Omega) > 0$ is the first eigenvalue of the Laplace operator in Ω with zero Dirichlet data on $\partial\Omega$, and $F_1(x)$ is the corresponding positive and normalized eigenfunction. The constant $c > 0$ is determined as the $L^2(\Omega)$-projection of u_0 on F_1.

Scaling techniques for the PME. The Cauchy Problem

In the case $m > 1$ the behaviour of our class of solutions can be described for large t by a one-parameter family of special solutions $\mathcal{U}(x,t;C)$ given by formula (2.23). Moreover, for a given solution u, there is a correct choice of the constant $C = C(u_0)$ in this asymptotic result which agrees with the rule of mass equality:

$$\int_{\mathbb{R}^N} u(x,t)\,dx = \int_{\mathbb{R}^N} \mathcal{U}(x,t;C)\,dx. \tag{2.39}$$

It follows that

$$C = c(m, N)\, M^{2(m-1)/[N(m-1)+2]}. \tag{2.40}$$

We also write $\mathcal{U}_M$ for the solution with mass M and F_M for its profile. This is the precise statement of the asymptotic convergence result:

Theorem 2.3 *Let $u(x,t)$ be the unique weak solution of the Cauchy problem with initial data $u_0 \in L^1(\mathbb{R}^N)$, $u_0 \geq 0$. Let $\mathcal{U}_M$ be the ZKB solution with the same mass as u_0. Then as $t \to \infty$ we have*

$$\lim_{t\to\infty} \|u(t) - \mathcal{U}_M(t)\|_1 = 0. \tag{2.41}$$

Convergence holds also in L^∞-norm in the proper scale:

$$\lim_{t\to\infty} t^{\alpha} \|u(t) - \mathcal{U}_M(t)\|_\infty = 0 \tag{2.42}$$

with $\alpha = N/[N(m-1)+2]$. Moreover, for every $p \in (1,\infty)$ we have

$$\lim_{t\to\infty} t^{\alpha(p-1)/p} \|u(t) - \mathcal{U}_M(t)\|_{L^p(\mathbb{R}^N)} = 0. \tag{2.43}$$

The last result follows from (2.41) and (2.42) by simple interpolation, but (2.42) and (2.41) are (to an extent) independent.

A proof. We will take from the text [307] the main ideas of proof of this theorem. We will follow the "four-step method", a general plan to prove asymptotic convergence devised by S. Kamin and Vazquez in 1988, [210], who settled in this way the asymptotic behaviour both for the p-Laplacian equation, $u_t = \nabla\cdot(|\nabla u|^{p-2}\nabla u)$, and for the PME. But the first proof of convergence for the PME in several dimensions appeared in a celebrated paper by A. Friedman and Kamin in 1980 [123]: it uses a method of optimal lower barriers that we will not present at this stage and obtain a weaker version of the result; the reader can consult the original paper or [307]. A previous proof in one space dimension is due to Kamin in 1973 [203].

Step 1a. RESCALING. In order to observe the asymptotic behaviour of the orbit of the Cauchy problem we rescale it according to the exponents of the ZKB solution. Let us see the whole story of scaling transformations in some detail for the reader's convenience. We consider a solution $u = u(x,t) \geq 0$ of (2.2) in the class of strong solutions with finite mass introduced in Section 2.2. We apply the group of dilations in all the variables

$$u' = Ku, \qquad x' = Lx, \qquad t' = Tt, \tag{2.44}$$

and impose the condition that when u' is expressed as a function of x' and t', i.e.,

$$u'(x',t') = Ku\left(x'/L, t'/T\right), \tag{2.45}$$

it is again a solution of (2.2). Then

$$\frac{\partial u'}{\partial t'} = \frac{K}{T}\frac{\partial u}{\partial t}\left(\frac{x'}{L}, \frac{t'}{T}\right), \qquad \Delta_{x'}(u')^m = \frac{K^m}{L^2}\Delta_x(u^m)\left(\frac{x'}{L}, \frac{t'}{T}\right).$$

Hence, (2.45) will be a solution if and only if $KT^{-1} = K^m L^{-2}$, i.e.,

$$K^{m-1} = L^2 T^{-1}. \tag{2.46}$$

We thus obtain a two-parameter transformation group acting on the set of solutions of (2.2). Choosing as free parameters L and T, it can be written as

$$u'(x', t') = L^{\frac{2}{m-1}} T^{-\frac{1}{m-1}} u(x, t) = \left(L^2/T\right)^{\frac{1}{m-1}} u\left(x'/L, t'/T\right).$$

Using standard letters for the independent variables and putting $u' = \mathcal{T}u$, we get

$$(\mathcal{T}u)(x, t) = L^{\frac{2}{m-1}} T^{-\frac{1}{m-1}}\, u\left(x/L, t/T\right). \tag{2.47}$$

This is just another way of writing the scaling law (2.31). Note that we have two degrees of freedom, which is too much for our purposes. The way the extra parameter is eliminated depends on the particular problem and is a very delicate question in the application of the scaling technique to asymptotic problems.

1b. The solution of the indeterminacy in our case is to use one of the parameters to force the scaling operator $\mathcal{T}$ to preserve some important behaviour of the orbit. Here we recall that $\mathcal{U}_M(x, t)$ has a constant mass; actually, this characterizes uniquely the solution (which is the ideal orbit we want to approach). Imposing thus the condition of mass conservation at $t = 0$, we get

$$\int_{\mathbb{R}^N} (\mathcal{T}u_0)(x)\, dx = \int_{\mathbb{R}^N} u_0(x)\, dx, \tag{2.48}$$

namely,

$$\int_{\mathbb{R}^N} K\, u_0\left(\frac{x}{L}\right)\, dx = \int_{\mathbb{R}^N} u_0(x)\, dx.$$

It easily follows that $K\, L^N = 1$. This and (2.46) give the expressions

$$K = T^{-\alpha}, \qquad L = T^{\beta}, \tag{2.49}$$

with the exponents (2.24). The transformation we are going to use is finally

$$(\mathcal{T}u)(x, t) = T^{-\alpha} u\left(x/T^{\beta}, t/T\right).$$

It is convenient to write the scaling factor in terms of $\lambda = 1/T$. Then, the solution is

$$\widetilde{u}_\lambda(x, t) = (\mathcal{T}_\lambda u)(x, t) = \lambda^\alpha u(\lambda^\beta x, \lambda t) \tag{2.50}$$

with initial data $\widetilde{u}_{0,\lambda}(x) = (\mathcal{T}_\lambda u_0)(x) = \lambda^\alpha u_0(\lambda^\beta x)$. This is the scaling formula that we call the λ-scaling or *fixed scaling*. It is in fact a family of scalings with free parameter $\lambda > 0$, that performs a kind of *zoom* on the solution. To end this step we note the following important property: the source-type solutions are invariant under the λ-rescaling, i.e., $\mathcal{U}_M(t) = \mathcal{T}_\lambda(\mathcal{U}_M(t))$.

Step 2a. UNIFORM ESTIMATES. We want to show that the family $\{\widetilde{u}_\lambda(t), \lambda > 0\}$ is uniformly bounded and even relatively compact in suitable functional spaces. This is an important step where we put to work the estimates derived in Section 2.2. It is crucial that the rescaling performed in the previous step and the estimates match, otherwise this step could not work.

To begin with our case, the family is uniformly bounded in $L^1(\mathbb{R}^N)$ for t positive:

$$\int_{\mathbb{R}^N} \widetilde{u}_\lambda(x,t)dx = \int_{\mathbb{R}^N} \lambda^\alpha u(\lambda^\beta x, \lambda t)\,dx = \int_{\mathbb{R}^N} u(y, \lambda t)\,dy = M < \infty. \quad (2.51)$$

Using now (2.25), we get

$$\|\widetilde{u}_\lambda(\cdot,1)\|_\infty = \lambda^\alpha \|u(\cdot,\lambda)\|_\infty \le \lambda^\alpha \frac{M^{2\alpha/N}}{\lambda^\alpha} C = CM^{2\alpha/N} \quad (2.52)$$

independently of λ, and in the same way

$$\|\widetilde{u}_\lambda(\cdot,t_0)\|_\infty = \lambda^\alpha \|u(\cdot,t_0\lambda)\|_\infty \le \lambda^\alpha \frac{M^{2\alpha/N}}{\lambda^\alpha t_0^\alpha} C = C\,M^{2\alpha/N}\,t_0^{-\alpha}. \quad (2.53)$$

Control of the norms $\|\cdot\|_1$ and $\|\cdot\|_\infty$ means control of all norms $\|\cdot\|_p$ for all $p \in [1,\infty]$. Thus, $\|\widetilde{u}_\lambda(\cdot,t)\|_p$ are equi-bounded for all $p \in [1,\infty]$.

Next, we take $t_0 > 0$ so that, by the regularizing effect, $u(t_0) \in L^{m+1}(\mathbb{R}^N)$. The energy estimates of Section 2.2, Property 9, give

$$\int_{\mathbb{R}^N} |\nabla \widetilde{u}_\lambda^m(x,t)|^2 dx \le C(t_0, \|\widetilde{u}_\lambda(\cdot,t_0)\|_{L^{m+1}}) \quad (2.54)$$

for $t \ge t_0 > 0$. Now, the $\|\widetilde{u}_\lambda(t)\|_{L^{m+1}}$ are equi-bounded, hence $\|\nabla \widetilde{u}_\lambda^m(\cdot,t)\|_{L^2}$ are equi-bounded (for $t \ge t_0 > 0$). Moreover, by (2.30) of Property 10, Section 2.2,

$$\|\frac{\partial \widetilde{u}_\lambda}{\partial t}(t)\|_{L^1} \le C\frac{\|\widetilde{u}_\lambda(t)\|_{L^1}}{t}. \quad (2.55)$$

Using the same argument, we conclude that the norms $\|(\widetilde{u}_\lambda)_t(t)\|_{L^1}$ are equi-bounded if $t \ge t_0 > 0$.

2b. COMPACTNESS. Let us recall the Rellich–Kondrachov theorem. *Let Ω be a bounded domain with C^1 boundary. Then*

$p < N \implies W^{1,p}(\Omega) \subset L^q(\Omega)$ *for all* $q \in [1, p^*)$, $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}$,
$p = N \implies W^{1,p}(\Omega) \subset L^q(\Omega)$ *for all* $q \in [1, +\infty)$,
$p > N \implies W^{1,p}(\Omega) \subset C(\overline{\Omega})$.

All these injections are compact. In particular, $W^{1,p}(\Omega) \subset L^p(\Omega)$ *with compact injection for all* $p \ge 1$. *In* $\Omega = \mathbb{R}^N$, *the above injections are compact in local topology (convergence on compact subsets).*

Let us now recall our situation for the family $\{\widetilde{u}_\lambda\}_{\lambda>1}$ for $t \ge t_0 > 0$:

$$\widetilde{u}_\lambda(x,t) \in L^\infty_{x,t} \subset L^1_{\text{loc}}, \quad \frac{\partial \widetilde{u}_\lambda}{\partial t}(x,t) \in L^\infty_t(L^1_x) \subset L^1_{x,t}\ (t \in (t_0,t_1)),$$

and

$$\nabla_x \widetilde{u}^m_\lambda \in L^2_{x,t} \subset L^1_{x,t}.$$

All spaces in time are local in the sense that they exclude $t = 0$. Here is the conclusion of this step:

Lemma 2.4 *The family $\{\widetilde{u}_\lambda\}_{\lambda>1}$ is relatively compact locally in $L^1_{x,t}$. Also the family $\{\widetilde{u}^m_\lambda\}_{\lambda>1}$.*

Step 3. PASSAGE TO THE LIMIT. We can now take a sequence $\{\lambda_k\} \to \infty$ and assert that $\widetilde{u}_{\lambda_k}$ converges in $L^1_{\text{loc}}(Q)$ to some function U:

$$\lim_{\lambda\to\infty} \widetilde{u}_\lambda(x,t) = U(x,t). \tag{2.56}$$

We need to study the properties of such *limit functions* $U(x,t)$.

Lemma 2.5 *Any limit U is a nonnegative weak and strong solution of* (2.2) *satisfying uniform bounds in $L^1(\mathbb{R}^N)$ and $L^\infty(\mathbb{R}^N)$ for all $t \ge \tau > 0$.*

Proof. It is clear that, as a consequence of the passage to the limit, U is nonnegative. Also, $U(t)$ is uniformly bounded in L^1 and L^∞ for $t \ge t_0 > 0$, according to formulas (2.22), (2.25). In order to check that it is a weak solution, we review the sense in which $\widetilde{u}_\lambda$ is a weak solution:

$$\iint \{\widetilde{u}_\lambda \varphi_t - \nabla(\widetilde{u}^m_\lambda) \cdot \nabla\varphi\}\, dx\, dt + \int \widetilde{u}_{0\lambda}(x)\varphi(x,0)\, dx = 0$$

for all φ tested. We have already remarked that our uniform estimates are not good near $t = 0$. In view of this, we restrict the test functions to the class

$$\varphi \in C^\infty_0(\mathbb{R}^N \times (0,\infty)),$$

so that φ vanishes in a neighborhood of $t = 0$. Then

$$\iint \{\widetilde{u}_\lambda \varphi_t - \nabla\widetilde{u}^m_\lambda \cdot \nabla\varphi\} dx\, dt = 0. \tag{2.57}$$

With our estimates

$$\begin{cases} \widetilde{u}_\lambda \to U & \text{locally in } L^1_{x,t}, \\ \widetilde{u}_\lambda \to U & \text{weak* in } L^\infty_{\text{loc}}, \\ \nabla\widetilde{u}^m_\lambda \to \nabla U^m & \text{in } L^2_{x;t,\text{loc}} \text{weak}, \end{cases}$$

we may pass to the limit in this expression (along a subsequence $\{\lambda_k\} \to \infty$) to get

$$\iint \{U\varphi_t - \nabla_x U^m \cdot \nabla_x\varphi\}\, dx\, dt = 0. \tag{2.58}$$

This means that U is a weak solution of equation (2.2). In fact, if $\tau > 0$,

$$\int_\tau^\infty \int_{\mathbb{R}^N} \{U\varphi_t - \nabla_x U^m \cdot \nabla_x \varphi\}\, dx\, dt + \int_{\mathbb{R}^N} U(x,\tau)\varphi(x,\tau)\, dx = 0.$$

Step 4. IDENTIFICATION OF THE LIMIT. Thus far, we have posed the dynamics in the form of an initial value problem and we have introduced a method of rescaling which has allowed us to obtain, after passage to the limit, one or several new solutions of the original problem. These solutions, which we call the asymptotic dynamics, form a special subset of the set of all orbits of our dynamical system and represent the (scaled) asymptotic behaviour of the original orbits. Their complete description becomes our main problem, a problem that may turn out to be difficult to solve.

In the present case the asymptotic dynamics turns out to be quite simple. We want to prove that the limit U along any sequence $\{\lambda_k\} \to \infty$ is necessarily $\mathcal{U}_M$. Both U and $\mathcal{U}_M$ are solutions of the PME for $t > 0$ enjoying a number of similar bounds. In order to identify them we only need to check their initial data and use a suitable uniqueness theorem for the Cauchy problem. The necessary uniqueness theorem is available thanks to M. Pierre's work [268].

Theorem 2.6 *Weak solutions of the PME in the class $u \in C((0,\infty) : L^1(\mathbb{R}^N))$, $u \geq 0$, which take a bounded and nonnegative measure $\mu(x)$ as initial data, i.e., such that*

$$\lim_{t\to 0} \int_{\mathbb{R}^N} u(x,t)\varphi(x)\, dx = \int_{\mathbb{R}^N} \varphi(x)\, d\mu(x), \tag{2.59}$$

for all $\varphi \in C_b(\mathbb{R}^N)$, $\varphi \geq 0$, are uniquely determined by the initial measure.

With $C_b(\mathbb{R}^N)$ we denote the space of continuous and bounded functions in $\mathbb{R}^N$. Let us then worry about the initial data. At first sight it looks easy:

Lemma 2.7 *If $\lambda \to \infty$, then $\lim \widetilde{u}_{0,\lambda}(x) \to M\delta(x)$ in the sense of bounded measures.*

Proof. As $\lambda \to \infty$, since $\alpha = N\beta > 0$,

$$\int_{\mathbb{R}^N} \widetilde{u}_{0\lambda}(x)\, \varphi(x)\, dx = \int_{\mathbb{R}^N} \lambda^\alpha u_0(\lambda^\beta x)\varphi(x)\, dx = \int_{\mathbb{R}^N} u_0(y)\varphi(y/\lambda^\beta)\, dy,$$

which converges to $\int_{\mathbb{R}^N} u_0(y)\, \varphi(0)\, dy$ for all $\varphi \in C_0^\infty(\mathbb{R}^N)$, $\varphi \geq 0$. We have used the mass value: $\int_{\mathbb{R}^N} u_0(y)\, dy = M$. □

The problem of the double limit. Unfortunately, the fact that the initial data for $\widetilde{u}_\lambda$ converge to $M\,\delta(x)$ does not justify by itself that $U(t)$ takes initial data $M\,\delta(x)$, because we do not control the evolution of the $\widetilde{u}_\lambda$ near $t = 0$ in a uniform way and a discontinuity might be taking place near $t = 0$ in the limit $\lambda \to \infty$. This is a typical case of double limits: does

$$\lim_{t\to 0}\lim_{\lambda\to\infty} \widetilde{u}_\lambda(x,t) = \lim_{\lambda\to\infty}\lim_{t\to 0} \widetilde{u}_\lambda(x,t)\,?$$

Preparing for a correct analysis, the first thing to do is to check that U and $\mathcal{U}_M$ have the same mass, i.e., that U has mass M. Since

$$\int_{\mathbb{R}^N} \widetilde{u}_\lambda(x,t)\,dx = M,$$

and $\widetilde{u}_{\lambda_k}$ converges to U in $L^1_{x,t}$-strong locally, we have $\widetilde{u}_{\lambda_k}(t) \to U(t)$ for a.e. t in $L^1_x(\mathbb{R}^N)$ locally and a.e. in $(x,t) \in Q$. By Fatou's lemma

$$\int_{\mathbb{R}^N} U(t)\,dx \le \lim_{k\to\infty} \int_{\mathbb{R}^N} \widetilde{u}_{\lambda_k}(x,t)\,dx = M,$$

hence the mass is equal or less. We have met again a difficulty. This difficulty is in principle essential. There are examples for rather simple equations in the nonlinear parabolic area where the initial data are not trivial but the whole solution disappears in the limit! Should such a "disaster" happen, we refer to it as an *initial layer of discontinuity*, an interesting object of study.

Compactly supported solutions. Here the only way the discontinuity can happen is by mass escaping to infinity, since there is only one mechanism at play, diffusion. In view of this difficulty we change tactics and try to establish the result under an extra assumption: *we take u_0 a bounded, $0 \le u_0 \le C$, and compactly supported function,* $\operatorname{supp}(u_0) \subset B_R(0)$.

Then, $\operatorname{supp}(\widetilde{u}_{0\lambda}) \subset B_{R/\lambda^\beta}(0)$. Moreover, there exists a source-type solution of the form $V(x,t) = \mathcal{U}_{M'}(x,t+1)$ with $M' \gg M$ such that $V(x,0) = \mathcal{U}_{M'}(x,1) \ge u_0(x)$. Then,

$$\widetilde{u}_\lambda(x,0) = \lambda^\alpha u_0(x\lambda^\beta,0) \le \lambda^\alpha \mathcal{U}_{M'}(x\lambda^\beta,1) = \mathcal{U}_{M'}\Big(x,\frac{1}{\lambda}\Big),$$

where in the last equality we have used the invariance of $\mathcal{U}$ under $\mathcal{T}_\lambda$. We conclude from the maximum principle that

$$\widetilde{u}_\lambda(x,t) \le \mathcal{U}_{M'}\Big(x,t+\frac{1}{\lambda}\Big), \tag{2.60}$$

and in the limit $U(x,t) \le \mathcal{U}_{M'}(x,t)$. The bound solves all our problems since it implies that the support of the family $\{\widetilde{u}_\lambda(t)\}$ is uniformly small for all λ large and t close to zero. Indeed, we observe the relation between the radii of the supports of a solution and its rescaling:

$$R_\lambda(t) = \frac{1}{\lambda^\beta} R(\lambda t). \tag{2.61}$$

It follows that the support of $\widetilde{u}_\lambda(t)$ is contained in a ball of radius

$$R = C\,(M')^{(m-1)\beta}\Big(t+\frac{1}{\lambda}\Big)^\beta \tag{2.62}$$

with $C = C(m,N)$. Now we can proceed.

Lemma 2.8 *The limit U has mass M for all $t > 0$.*

This is a consequence of the dominated convergence theorem since U is bounded above by a big source-type (ZKB) solution.

Lemma 2.9 *Under the present assumptions on u_0, we have $U(x,t) \to M\delta(x)$ as $t \to 0$, i.e.,*

$$\lim_{t\to 0} \int_{\mathbb{R}^N} U(x,t)\varphi(x)dx = M\varphi(0) \tag{2.63}$$

for all test functions $\varphi \in C_0^\infty(\mathbb{R}^N)$.

Proof. Since $M = \int U(x,t)\,dx$, we have for $t > 0$,

$$|\textstyle\int [U(x,t)\varphi(x) - M\varphi(0)]\,dx| \le \int |U(x,t)||\varphi(x) - \varphi(0)|dx$$
$$\le \textstyle\int_{|x|\le\delta} |U(x,t)||\varphi(x) - \varphi(0)|dx + \int_{|x|>\delta} |U(x,t)||\varphi(x) - \varphi(0)|dx = (*).$$

By continuity there exists $\delta > 0$ such that $|\varphi(x) - \varphi(0)| \le \varepsilon/2M$ if $|x| \le \delta$. Besides, φ is bounded so that

$$|\varphi(x) - \varphi(0)| \le 2C \quad (\varphi \in C_0^\infty)\,.$$

Since $U(x,t)$ vanishes for $|x| \ge \delta$ if t is small enough, we get

$$(*) \le M\frac{\varepsilon}{2M} + 2C\int_{|x|>\delta} |U(x,t)|\,dx \le K\varepsilon\,.$$

Conclusion. Using the uniqueness result, Theorem 2.6, we can identify U. Hence, for $t = 1$ we have $\widetilde{u}_{\lambda_k}(x,1) \to \mathcal{U}_M(x,1)$ in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$. Now, the $\widetilde{u}_\lambda$ have compact support which is uniformly bounded in λ. It follows that

$$\widetilde{u}_{\lambda_k}(x,1) \to \mathcal{U}_M(x,1) \qquad \text{in } L^1\text{-strong.}$$

(We pass from local to global convergence.) The limit is thus independent of the sequence $\{\lambda_k\}$. It follows that the whole family $\{\widetilde{u}_\lambda\}$ converges to $\mathcal{U}_M$ as $\lambda \to \infty$.

General data. We still have to deal with data which do not have compact support. The proof in this case implies some nontrivial extra effort for which we refer to [307]. Such type of extra difficulty is quite typical of similar problems.

Step 5. REPHRASING THE RESULT. The argument has concluded, but we still have to write the conclusion in the original variables and scales. So actually the "4-step method" is rather a "5-step method", having a simple end step. Let $F_M(x) = \mathcal{U}_M(x,1)$. We have just proved that

$$\lim_{\lambda\to\infty} \|\lambda^\alpha u(\lambda^\beta x, \lambda) - F_M(x)\|_{L^1} = 0,$$

which means with $y = \lambda^\beta x$ that

$$\lim_{\lambda\to\infty}\int \lambda^{\alpha}|u(y,\lambda)-\lambda^{-\alpha}F_M(y/\lambda^{\beta})|\lambda^{-\beta N}\,dy=0.$$

Noting that $\mathcal{U}_M(y,\lambda)=\lambda^{-\alpha}F_M(y/\lambda^{\beta})$ and that $\alpha=\beta N$, we arrive at

$$\lim_{\lambda\to\infty}\int |u(y,\lambda)-\mathcal{U}_M(y,\lambda)|\,dy=0,$$

i.e., replacing λ by t,

$$\lim_{t\to\infty}\|u(y,t)-\mathcal{U}_M(y,t)\|_{L^1_y}=0.$$

This is the asymptotic formula (2.41).

The continuous scaling version. A different way of implementing the scaling of the orbits of the Cauchy problem and proving the previous facts consists of using *continuous rescaling*, which in this case is written in the form

$$\theta(\eta,\tau)=t^{\alpha}u(x,t),\qquad \eta=x\,t^{-\beta},\qquad \tau=\ln t, \tag{2.64}$$

with α and β the standard similarity exponents given by (2.24). Then t^{α} and t^{β} are called the *scaling factors* (or *zoom factors*), while τ is called the *new time*. With respect to the λ-scaling, we see that the zoom factors change continuously with time, hence the name. We may also call it *time-adapted* rescaling.

This version of the scaling technique has a very appealing dynamical flavor and it will appear often in the sequel. The reader should note that *every problem has its corresponding zoom factors that have to be determined as a part of the analysis.*

In our problem, the new orbit $\theta(\tau)$ satisfies the equation

$$\theta_{\tau}=\Delta(\theta^m)+\beta\eta\cdot\nabla\theta+\alpha\theta. \tag{2.65}$$

It is bounded uniformly in $L^1(\mathbb{R}^N)\cap L^{\infty}(\mathbb{R}^N)$. The source-type (ZKB) solutions transform into the stationary profiles F_M in this transformation, i.e., $F(\eta)$ solves the nonlinear elliptic problem

$$\Delta f^m+\beta\eta\cdot\nabla f+\alpha f=0. \tag{2.66}$$

The boundedness and compactness arguments developed before apply here, and we may pass to the limit and form the ω-limit, which is the set

$$\omega(\theta)=\{f\in L^1(\Omega):\exists\ \{\tau_j\}\to\infty \text{ such that } \theta(\tau_j)\to f\}. \tag{2.67}$$

The convergence takes place in the topology of the functional space in question, here any $L^p(\Omega)$, $1\le p<\infty$ (strong).

The rest of the proof consists in showing that the ω-limit is just the ZKB profile F_M. The argument can be translated in the following way. Corresponding to the sequence of scaling factors λ_k of the previous subsection, we take a sequence of *delays* $\{s_k\}$ and define

$$\theta_k(\eta, \tau) = \theta(\eta, \tau + s_k). \tag{2.68}$$

The family $\{\theta_k\}$ is precompact in $L^\infty_{\text{loc}}(\mathbb{R}_+ : L^1(\mathbb{R}^N))$; hence, passing to a subsequence if necessary, we have

$$\theta_k(\eta, \tau) \to \widetilde{\theta}(\eta, \tau). \tag{2.69}$$

Again, it is easy to see that $\widetilde{\theta}$ is a weak solution of (2.65) satisfying the same estimates. The end of the proof identifies it as a stationary solution, which was done in the previous proof by the other scaling method, the *fixed rescaling.* Theorem 2.3 can now be used to characterize the stationary solutions.

Theorem 2.10 *The profiles F_M can be characterized as the unique solution of equation* (2.66) *such that $f \in L^1(\mathbb{R}^N)$, $f^m \in L^1_{\text{loc}}(\mathbb{R}^N)$ and $f \geq 0$. The conditions $f^m \in H^1(\mathbb{R}^N)$, $f \in C(\mathbb{R}^N)$ are true, but not needed in the proof.*

Proof. Any other solution f can be taken as initial data for the evolution equation (2.65) and then Theorem 2.3 proves that the corresponding solution of (2.65) converges to the source-type solution with the same mass, F_M. Now, the solution $u(x,t) = t^{-\alpha} f(x\, t^{-\beta})$ is an admissible solution of the PME which converges in the rescaling to f. Therefore, $f = F_M$. □

The fast diffusion case. The extension of the asymptotic properties proved above to exponent $m = 1$ gives as a consequence results that are well known for the classical heat equation. It is interesting to remark that the proof given here applies (with inessential minor changes) and is very different from the usual proof, based on the representation formula.

We can even go below $m = 1$ and prove similar results for some so-called fast-diffusion equations, i.e., equation (2.2) with $0 < m < 1$. To start with we need two basic ingredients.

(a) A theory of well-posedness for the Cauchy problem. As we have said, the results of Section 2.2 apply also in this case with minor easy changes. The main novelty is that solutions are positive everywhere and C^∞-smooth, which is rather good news in this context.

(b) The second ingredient is the model of asymptotic behaviour. The source-type solution exists just for $m > m_c = (N-2)_+/N$ and it can be conveniently written in the form

$$\mathcal{U}_M(x,t) = \left(\frac{Ct}{|x|^2 + At^{2\beta}}\right)^{1/(1-m)} = \frac{K\, t^{-\alpha}}{[A + (|x|\, t^{-\beta})^2]^{1/(1-m)}}, \tag{2.70}$$

where $\beta = 1/[2 - N(1-m)]$ is positive precisely in that range, $\alpha = N\beta$, $C = 2m/\beta(1-m)$ is a fixed constant, $K = C^{1/(1-m)}$, and $A > 0$ is an arbitrary constant that can be determined as a decreasing function of the mass $M = \int \mathcal{U}(x,t)\, dx$, $A = k(m,N)\, M^{-\gamma}$ with $\gamma = 2(1-m)\beta$.

In dimensions $N = 1, 2$ the whole range $0 < m < 1$ is covered. However, the *critical exponent*, $m_c = 1 - (2/N)$, is larger than zero for $N \geq 3$. It is then proved that for $0 < m < m_c$ no solution of the ZKB type exists (i.e., self-similar with constant positive mass). The value $m_c = (N-2)/N$ is related to the Sobolev embedding exponents as the reader will easily realize.

CONVERGENCE IN RELATIVE ERROR. It can be checked that the convergence results of Theorem 2.3 hold true for $m > m_c$, and the proofs given above are true but for minor details. However, the fact that the solutions of the fast diffusion equation do not have the property of conserving compact supports, but rather develop tails at infinity of a certain form gives rise to a very interesting estimate formulated in terms of *relative error*, or in other words, as *weighted convergence*, that we present next. It requires a suitable behaviour of the initial data as $|x| \to \infty$ (similar in decay to the ZKB solution).

Theorem 2.11 *Under the assumption that u_0 is bounded and $u_0(x) = O(|x|^{-\frac{2}{1-m}})$ as $|x| \to \infty$, we have the asymptotic estimate*

$$\lim_{t\to\infty} \frac{|u(x,t) - \mathcal{U}(x,t;M)|}{\mathcal{U}(x,t;M)} \to 0 \tag{2.71}$$

uniformly in $x \in \mathbb{R}^N$. The condition on the initial data can be weakened into the integral estimate

$$\int_{|y-x|\le|x|/2} |u_0(y)|\,dy = O(|x|^{N-\frac{2}{1-m}}) \quad \text{as } |x| \to \infty. \tag{2.72}$$

In particular, we have $\|u(t) - \mathcal{U}(t;M)\|_1 \to 0$ as $t \to \infty$ (as in case $p = 1$ of Theorem 2.3), and $t^\alpha|u(x,t) - \mathcal{U}(x,t;M)| \to 0$ as $t \to \infty$ uniformly in x (case $p = \infty$), but estimate (2.71) is much more precise because the convergence is uniform with weight

$$\rho = (|y|^2 + c)^{1/(1-m)}, \qquad y = x\,t^{-\beta}.$$

For the detailed proof we refer to [307].

The cases $m \le m_c$ have different asymptotics. We will return to the fast diffusion equation with critical exponent $m_c = (N-2)/N$ in Chapter 6 as a case of matched asymptotics. And we recall that the range $0 < m < (N-2)/N$ is still partially understood, though it is known that a unique second-kind similarity solution with finite time extinction is stable, [152]; see Chapter 6. This should allow the reader to get an idea of the difficulty of the problems of nonlinear asymptotic analysis.

Other problems

We will see in the text that many of these ideas can be applied to different problems, and in doing so there appear different variants. A closely related case with minimal difficulty is the initial and boundary value problem for the PME equation in a bounded domain $\Omega \subset \mathbb{R}^N$ with zero boundary data. In that case the asymptotic model is not a self-similar solution of the ZKB type, but rather a separated-variables solution of the form

$$\mathcal{U}(x,t) = (t+c)^{-\alpha} F(x), \tag{2.73}$$

where $\alpha = 1/(m-1)$ and $c \geq 0$ is a free constant. By default we put $c = 0$. The profile $F \geq 0$ is calculated as the unique nontrivial solution of the nonlinear elliptic problem

$$\Delta F^m + \alpha F = 0 \quad \text{in } \Omega, \qquad F = 0 \quad \text{on } \partial\Omega.$$

The continuous scaling is given by

$$u(x,t) = t^{-\alpha}\theta(x,\tau), \quad \tau = \ln t. \tag{2.74}$$

Then θ satisfies the nonlinear reaction-diffusion equation

$$\theta_\tau = \Delta\theta^m + \alpha\theta \tag{2.75}$$

which is autonomous, i.e., time does not appear explicitly. Observe that the new time τ ranges from $-\infty$ to ∞. The initial time $t = 0$ corresponds to $\tau = -\infty$, but displacing the origin of time t allows us to take any finite initial time for τ, like $\tau_0 = 0$ if the reader feels more comfortable. The location of the time origin does not alter the asymptotic problem and is then a question of convenience; precisely for this reason many authors use a slightly different definition, $t + 1 = e^\tau$, which makes $t = 0$ equivalent to $\tau = 0$. Equation (2.75) would be the same. We take zero Dirichlet boundary data, in the sense that $\theta^m \in H_0^1(\Omega)$. The initial data are taken nonnegative and integrable in Ω. The possibility of delaying the time origin and the regularity theory allow us to assume that $\theta(x, 0)$ is bounded, even continuous.

Theorem 2.12 *There exists a unique nonnegative, nontrivial self-similar solution of the PME of the form* (2.73), *such that if u is any weak solution of the homogeneous Cauchy–Dirichlet problem, we have*

$$\lim_{t\to\infty} t^\alpha |u(x,t) - \mathcal{U}(x,t)| = \lim_{t\to\infty} |t^\alpha u(x,t) - F(x)| = 0 \tag{2.76}$$

uniformly in Ω.

2.4 The Lyapunov method

The work of A.M. Lyapunov in 1892 has had a lasting influence on the studies of stability not only for ordinary differential equations but also for general dynamical systems, and in particular for PDEs (which are infinite-dimensional dynamical systems). However, it must be noted that it is not always easy to find the way of applying Lyapunov's second method to nonlinear heat equations. Actually, one of the main points of the present book is to supply an alternative tool when Lyapunov methods are either nonexistent or difficult to apply.

We devote the next two subsections to derive alternative proofs of the main convergence result for the PME, Theorem 2.3. Both are based on standard implementations of the idea of *Lyapunov Function.*

Lyapunov function for the PME

I. Given an orbit $\{u(t)\}$ of the PME in the framework of Sections 2.2 and 2.3, having mass $M > 0$, we introduce the functional

$$J_u(t) = \int_{\mathbb{R}^N} |u(x,t) - \mathcal{U}_M(x,t)|\, dx. \tag{2.77}$$

It is clear from the contraction property that $J_u(t)$ is nonincreasing in t. We get the following result.

Lemma 2.13 *The limit $J_\infty = \lim_{t\to\infty} J_u(t) \ge 0$ exists.*

Note that $J_u(t)$ becomes zero only if $u(t)$ coincides with the ZKB solution for some $t_1 > 0$; then the equality holds for all $t \ge t_1$ and the asymptotic result is trivial. Otherwise $J_u(t) > 0$ for all $t > 0$. We have to examine this case.

II. LIMIT SOLUTIONS. We perform Steps 1, 2 and 3 of the preceding proof to obtain a sequence $\{\lambda_k\} \to \infty$ such that

$$\widetilde{u}_{\lambda_k}(x,t) \to U(x,t) \tag{2.78}$$

in $L^1(\mathbb{R}^N \times (t_1, t_2))$. The limit U is again a solution of the PME. It is nontrivial and has mass M (this is easy for compactly supported solutions and then true for the rest by approximation).

III. INVARIANCE PRINCIPLE. One of the key features of the use of Lyapunov functions is the following *Asymptotic Invariance Property*.

Lemma 2.14 *The Lyapunov function is constant on limit orbits, i.e., J_U does not depend on t.*

Proof. The Lyapunov function is translated to the rescaled family $\widetilde{u}_\lambda$ by the formula

$$J_{\widetilde{u}_\lambda}(t) = \int_{\mathbb{R}^N} |\widetilde{u}_\lambda(x,t) - \mathcal{U}_M(x,t)|\, dx = J_u(\lambda t). \tag{2.79}$$

It follows that for fixed $t > 0$, we have

$$\lim_{\lambda\to\infty} J_{\widetilde{u}_\lambda}(t) = \lim_{\lambda\to\infty} J_u(\lambda t) = J_\infty.$$

On the other hand, we see that J_u depends in a lower-semicontinuous form on u. Moreover, it is continuous under the passage to the limit that we have performed. Hence, for every $t > 0$, $J_U(t) = J_\infty$. □

IV. A LIMIT SOLUTION IS A SOURCE-TYPE SOLUTION. In order to identify U, the next result we need is the following.

Lemma 2.15 *Consider the orbit of $u(t)$ with mass $M > 0$ and with connected support for $t \ge t_0$. Then the function $J_u(t)$ is strictly decreasing in any time interval (t_1, t_2), $t_0 < t_1 < t_2$, unless $u = \mathcal{U}_M$ or both solutions have disjoint supports in that interval.*

Proof. We consider for $t \geq t_1 > 0$ the solution w of the PME with initial data at $t = t_1$,

$$w(x, t_1) = \max\{u(t_1), v(t_1)\}, \tag{2.80}$$

where we put $v = \mathcal{U}_M$ for easier notation. Clearly, $w \geq u$ and $w \geq v$, hence

$$w(t) \geq \max\{u(t), v(t)\}, \quad t > t_1.$$

Moreover, we have $w(x, t_1) - u(x, t_1) = (v(x, t_1) - u(x, t_1))_+$ and $w(x, t_1) - v(x, t_1) = (u(x, t_1) - v(x, t_1))_+$ so that

$$J_u(t_1) = \int_{\mathbb{R}^N} (w(t_1) - u(t_1))\, dx + \int_{\mathbb{R}^N} (w(t_1) - v(t_1))\, dx,$$

while for general $t > t_1$,

$$J_u(t) + 2\int_{\mathbb{R}^N} (w(t) - \max\{u(t), v(t)\})\, dx$$
$$= \int_{\mathbb{R}^N} (w(t) - u(t))\, dx + \int_{\mathbb{R}^N} (w(t) - v(t))\, dx.$$

Both integrals on the right-hand side are nonincreasing in time by the contraction principle, hence constancy of J_u in an interval $[t_1, t_2]$ implies that

$$w(t_2) = \max\{u(t_2), v(t_2)\}. \tag{2.81}$$

In order to examine the consequences of this equality we use the strong maximum principle.

Lemma 2.16 *Two ordered solutions of the PME cannot touch for $t > 0$ wherever they are positive.*

This is a standard result for classical solutions of quasilinear parabolic equations, cf. z[234]. It follows that (2.81) is then possible on any connected open set Ω where $w(\cdot, t_2) > 0$ under three circumstances:

(i) $w(t_2) = u(t_2) > v(t_2)$, or
(ii) $w(t_2) = v(t_2) > u(t_2)$, or
(iii) $w(t_2) = u(t_2) = v(t_2)$.

Since the support of the source-type solution is a ball and the support of u is also connected, we conclude the result of Lemma 2.15. □

Note. If M is not the mass of u, there is still another possibility for constant J_u, namely that the solutions are different but ordered: either $u(t) \geq \mathcal{U}_M(t)$ or $u(t) \leq \mathcal{U}_M(t)$.

We may now conclude the proof of Theorem 2.3 by this method in the case where u_0 has compact support, so that by standard properties of the propagation of support, it is connected after a certain time t_0. Since the source-type solution penetrates into the whole space eventually in time and U has a noncontracting support, it follows that for large t the supports of U and $\mathcal{U}_M$ do intersect. Since both solutions cannot be ordered because they have the same mass, $J_U(t)$ must be zero since it is not strictly decreasing by Lemma 2.14. We have thus proved that $J_\infty = 0$ and

$$U = \mathcal{U}_M, \tag{2.82}$$

which identifies all possible limits of rescalings as the unique source-type solution with the same mass. This ends the proof. The extension to general data is done by density, we omit the details. □

V. CONTINUOUS RESCALING. One way of proving the previous facts is by using the continuous rescaling, formula (2.64). As explained in the previous section, taking a sequence of delays $\{s_k\}$ we define

$$\theta_k(\eta, \tau) = \theta(\eta, \tau + s_k),$$

and passing to the limit yields

$$\theta_k(\eta, \tau) \to \widehat{\theta}(\eta, \tau). \tag{2.83}$$

Again it is easy to see that $\widehat{\theta}$ is a weak solution of (2.65) satisfying the same estimates. For θ the Lyapunov function is translated into

$$J_\theta(t) = \int_{\mathbb{R}^N} |\theta(\eta, \tau) - F_M(\eta)|\, d\eta, \tag{2.84}$$

and we see that it is continuous under the passage to the limit we have performed. Let us examine now the situation when $J_\infty > 0$. Then $\widehat{\theta} \neq F_M$ and the orbit of $\widehat{\theta}$ has a strictly decreasing functional, so that for $\tau_2 = \tau_1 + h$ we have

$$J_{\widehat{\theta}}(\tau_1) - J_{\widehat{\theta}}(\tau_2) = c > 0.$$

Since $\widehat{\theta}$ is the limit of the θ_k we get, for all large enough k,

$$J_{\theta_k}(\tau_1) - J_{\theta_k}(\tau_1 + h) \geq c/2.$$

But this means that for all k large enough,

$$J_\theta(\tau_1 + s_k) - J_\theta(\tau_1 + s_k + h) \geq c/2.$$

This contradicts the fact that J_θ has a limit. The proof is complete. □

Comment. As we had announced, the proof of this section uses several steps of the former with a completely different end. It contains some fine regularity results that can make it difficult to apply in more general settings. However, some of these difficulties can be overcome by other means. Lasalle's invariance principle is a powerful tool in dynamical systems [236], worth knowing also in this context.

Another Lyapunov approach for the PME

A different Lyapunov approach is based on the existence of the so-called *Newman functional* that can be written for continuously rescaled variables as

$$\mathcal{J}_\theta(\tau) = \int_{\mathbb{R}^N} \{\theta^m(\eta,\tau) + \kappa\,|\eta|^2\theta(\eta,\tau)\}\,d\eta\,, \qquad \kappa = \tfrac{1}{2}\beta(m-1)\,, \tag{2.85}$$

where β is the similarity exponent. The proof of convergence in this instance will be based on the possibility of calculating the value of $d\mathcal{J}_\theta/d\tau$ along an orbit.

Lemma 2.17 *Let* $\mathcal{J}_\theta$ *be the functional* (2.85). *Then for every rescaled orbit of the Cauchy problem, we have*

$$\frac{d\mathcal{J}_\theta}{d\tau} = -\frac{m^2}{m-1}\int \theta\,|\nabla(\theta^{m-1} + \frac{k}{m}\,|\eta|^2)|^2\,d\eta. \tag{2.86}$$

Proof. In order to analyze the evolution of $\mathcal{J}$ let us put for a moment

$$\mathcal{J}(\tau) = \int_{\mathbb{R}^N} \{\theta^m(\eta,\tau) + \lambda\,|\eta|^2\theta(\eta,\tau)\}\,d\eta,$$

with $\lambda > 0$. Let us perform the following formal computations:

$$\begin{aligned} d\mathcal{J}/d\tau &= \int (m\theta^{m-1} + \lambda\,|\eta|^2)\,\theta_\tau\,d\eta \\ &= \int (m\theta^{m-1} + \lambda\,|\eta|^2)(\Delta\theta^m + \beta\nabla\cdot(\eta\,\theta))\,d\eta \\ &= -\int \nabla(m\theta^{m-1} + \lambda\,|\eta|^2)\cdot(\nabla\theta^m + \beta\eta\theta)\,d\eta \\ &= -\int \theta\,\nabla(m\theta^{m-1} + \lambda\,|\eta|^2)\cdot\nabla(\frac{m}{m-1}\theta^{m-1} + \frac{\beta}{2}|\eta|^2)\,d\eta. \end{aligned}$$

In case $\lambda = \beta(m-1)/2$ we can write this quantity as (2.86), which proves that $\mathcal{J}_\theta$ is a Lyapunov function, i.e., it is monotone along orbits.

These computations are easily justified for classical solutions which decay quickly at infinity. The result for general solutions is then justified by a density argument using the regularity of the solutions of the PME, cf. [298] (but we can also restrict the Lyapunov analysis to the above mentioned class of solutions since the proof of convergence for general solutions is then completed by a density argument).

LIMIT ORBITS AND INVARIANCE. As in the previous section, we pass to the limit along sequences $\theta_k(\tau) = \theta(\tau+s_k)$ to obtain limit orbits $\widehat{\theta}(\tau)$, on which the Lyapunov function is constant, hence $d\mathcal{J}_{\widehat{\theta}}/d\tau = 0$.

IDENTIFICATION STEP. The proof of asymptotic convergence concludes in the present instance in a new way, by analyzing when $d\mathcal{J}_\theta/d\tau$ is zero. Here is the crucial observation that ends the proof: *the second member of* (2.86) *vanishes if and only if* θ *is a ZKB profile.* □

The rate of convergence can be calculated by computing $d^2\mathcal{J}_\theta/d\tau^2$, which is not easy.

2.5 Comparison techniques

We have already discussed a few applications of the standard comparison principle which guarantees the usual comparison of two solutions of the PME with ordered initial and boundary data. Such *barrier* techniques play a key role in the general theory of nonlinear uniformly parabolic equations. We have seen that the comparison principle remains valid in appropriate classes of weak (or strong) solutions to the PME and this holds as a general principle for the types of nonlinear equations listed at the beginning of the chapter.

We next discuss two other comparison techniques which are somehow related to each other (but not equivalent) and will play important parts in the sequel. Both are quite useful in dealing with evolution problems in one space dimension (or in several under conditions of radial symmetry), but no equivalent tool has been found to deal with problems in several dimensions. This will have as a consequence a real delay in the general development of the N-dimensional theory with respect to 1D or radial N-D.

2.5.1 Intersection comparison and Sturm's theorems

The historical origin of the nowadays well-known Intersection Comparison techniques is quite remarkable. In 1836 C. Sturm published two pioneering papers in the first volume of J. Liouville's *Journal de Mathématiques Pures et Appliquées*. One of them [294], devoted to the study of zeros of solutions $u(x)$ of second-order ODEs

$$u'' + q(x)u = 0, \quad x \in \mathbb{R}, \tag{2.87}$$

immediately exerted a great influence on the general theory of ODEs. Sturm oscillation, comparison and separation theorems can be found in most textbooks on ODEs with various generalizations to other equations and systems of equations. Such theorems classify and compare zeros and zero sets $\{x \in \mathbb{R} : u(x) = 0\}$ of different solutions $u_1(x)$ and $u_2(x)$ of equation (2.87), or solutions of equations with different continuous ordered potentials $q_1(x) \geq q_2(x)$.

The second paper [295] dealt with the *evolution* analysis of zeros and zero sets (i.e., the sets $\{x : u(x,t) = 0\}$) for solutions $u(x,t)$ of partial differential equations of parabolic type, for instance,

$$u_t = u_{xx} + q(x)u, \quad x \in [0, 2\pi], \ t > 0, \tag{2.88}$$

supplied with Dirichlet boundary condition $u = 0$ at $x = 0$ and $x = 2\pi$, and given smooth initial data at $t = 0$. Two different Sturm results for PDEs like (2.88) are found and can be stated as follows:

- First Sturm Theorem: Nonincrease in time of the number of zeros (or sign changes) of solutions.
- Second Sturm Theorem: Classification of blow-up self-focusing formations and collapses of multiple zeros.

Both Sturm theorems are usually referred together as to the *Sturmian argument* on zero-set analysis.

It is curious that most of Sturm's PDE paper [295] was devoted to the second theorem on the striking "dissipativity" properties of the evolution of zeros of solutions of linear parabolic equations, where a detailed backward-forward continuation analysis of collapse of multiple zeros of solutions was performed (actually, it is a perfect example of a complete asymptotic theory for such a "singularity formation" and the "collapse of singularity" immediately afterwards). The result of the first theorem then was a straightforward consequence of the second one; see p. 431 in [295].

First Sturm Theorem. We begin with a general presentation of this classical Sturm result for smooth solutions of one-dimensional linear parabolic equations. Let D and J be open bounded intervals in $\mathbb{R}$. Consider in $S = D \times J$ a linear parabolic equation

$$u_t = a(P)u_{xx} + b(P)u_x + c(P)u, \quad \text{where } P \text{ denotes } (x, t). \tag{2.89}$$

Given a constant $\tau \in J$, we denote by ∂S_τ the parabolic boundary of the domain $S_\tau = S \cap \{t < \tau\}$, i.e., the lateral sides and the bottom of the boundary of S_τ. Given a solution u defined on S_τ, the positive and negative sets of u are defined as follows:

$$U^+ = \{P : \ P \in S_\tau, \ u(P) > 0\}, \quad U^- = \{P : \ P \in S_\tau, \ u(P) < 0\}. \tag{2.90}$$

A *component* of U^+ (or U^-) is a maximal open connected subset of U^+ (or U^-).

Definition. Given $t \in \overline{J}$, the *number of sign changes* of $u(x,t)$ at time t, denoted by $Z(t,u)$, is the (finite or infinite) number of components of $\{x \in D : \ u(x,t) \neq 0\}$. Alternatively, $Z(t,u)$ is the supremum over all natural k such that there exist k points from D, $x_1 < x_2 < \cdots < x_k$, satisfying

$$u(x_j, t) \cdot u(x_{j+1}, t) < 0 \quad \text{for all } j = 1, 2, ..., k-1.$$

This number has been also called the *lap number*.

Theorem 2.18 (FIRST STURM THEOREM ON SIGN CHANGES). *Let a, b, c be continuous, bounded and $a \geq \mu > 0$ in S. Let $u(x,t)$ be a solution of* (2.89) *in S which is continuous on $\overline{S}$.*

(i) Suppose that on ∂S_τ there are precisely n (respectively m) disjoint intervals where u is positive (resp. negative). Then U^+ (resp. U^-) has at most n (resp. m) components in S_τ and the closure of each component must intersect ∂S_τ in at least one interval.

(ii) The number of sign changes $Z(\tau, u)$ of $u(x, \tau)$ on D is not greater than the number of sign changes of u on ∂S_τ.

We have taken this statement from D.H. Sattinger's paper [287], 1969 (similar to K. Nickel's [253], 1962). These results admit natural extensions to the Cauchy problem or other problems in unbounded domains if, under necessary assumptions on initial-boundary data and functional setting, we can control intersections of the solutions at infinity. We refer to S. Angenent's paper [8] where a detailed proof is available. We omit further details and present other key references in Remarks and Comments on the Literature. *Intersection Comparison* via Sturm's theorem will be used in proving important estimates on solutions of nonlinear parabolic equations in this text. We will give further references to the results which are most important for our applications at the end of this chapter and later on in the appropriate places.

The PME: General principles of intersection comparison. Consider weak nonnegative solution $u(x,t)$ of the Cauchy problem for the one-dimensional PME

$$u_t = (u^m)_{xx} \quad \text{in } \mathbb{R}\times\mathbb{R}_+, \quad u(x,0) = u_0(x) \geq 0 \quad \text{in } \mathbb{R}. \tag{2.91}$$

We assume that u_0 is a bounded continuous compactly supported function so that we can start intersection comparison from the initial moment of time $t = 0$ without using a suitable regular approximation of, say, L^1 initial data.

We are going to use the first Sturm theorem in order to compare $u(x,t)$ with a subset B of some particular (mostly explicit) solutions of the same equations. Fortunately, we are given an excellent three-parameter subset of the ZKB-solutions (2.23)

$$B = \{\mathcal{U}(x-a, t-\tau; C), \quad a, \tau \in \mathbb{R}, \ C \in \mathbb{R}_+\}, \tag{2.92}$$

where a, τ are translation parameters in space and time, and C is the mass parameter (2.40). As we know, weak solutions u and of course $\mathcal{U}$ (for any $t > \tau$) are continuous in x functions, so that we can define the number of intersections as the number of sign changes of the difference $w(x,t) = u(x,t) - \mathcal{U}(x,t)$:

$$I(t,\mathcal{U}) = Z(t,w) \quad \text{for } t > \tau\,.$$

Subtracting the equations for u and $\mathcal{U}$, we obtain that the difference w satisfies a linear parabolic equation

$$w_t = (a(x,t)w)_{xx}, \tag{2.93}$$

where by Hadamard's formula

$$a = m\int_0^1 (\theta u + (1-\theta)\mathcal{U})^{m-1}d\theta \geq 0.$$

If it satisfies the conditions of the Sturm theorems for linear equations, then the number of intersections $I(t,\mathcal{U})$ of two solutions u and $\mathcal{U}$ of the parabolic equation (2.91) obeys the same properties as the number of sign changes of the difference w satisfying (2.93).

At this stage, the main feature of the intersection comparison technique consists in using the fact that the property of nonincrease in time holds with respect to *any* fixed solution $\mathcal{U}(x,t) \in B$. In the simplest case, we study the evolution of *tangency points* or *inflection points* defined as in standard calculus. In other words, intersection comparison with the set B means that we apply the Sturm theorem relative to an infinite number of different linearized parabolic equations. The main ingredient of such a geometric theory is to organize such an intersection comparison in the most effective way. Often, we need the subset B of particular solutions to be complete (sufficiently dense) in a suitable geometric setting in order to exhaust necessary spatial shapes of the more general solution $u(x, t)$ under consideration. We also need some continuity, monotonicity and compactness properties of the subset B to be defined and checked for a number of problems. Fortunately, (2.92) is a very wide subset of explicit solutions depending on three parameters $\{a, \tau, C\}$ and this is a rare opportunity occurring for nonlinear equations. Therefore, a lot of general properties of weak L^1-solutions can be proved by intersection comparison, and later on we present various approaches of using geometric techniques based on playing with multiparametric families of particular solutions.

The reader should observe that there is a weak point in the above speculations: for a given subset of the ZKB-solutions, equation (2.93) *is not* uniformly parabolic, hence Theorem 2.18 does not apply. Fortunately, we can show that this is not essential for our purpose, since weak solutions satisfy the important property of admitting *smooth approximations*. This fundamental property of the PME (and other equations discussed above, having weak solutions of limited regularity) can be formulated as follows: under given assumptions, the unique weak solution $u(x, t)$ of the Cauchy problem (2.91) can be constructed as the limit

$$u = \lim_{n \to \infty} u_n \tag{2.94}$$

of sequences $\{u_n\}$ of classical solutions to the PME (2.91) with regularized strictly positive initial data $\{u_{0n}\} \to u_0$ uniformly in $\mathbb{R}$ satisfying $u_{0n}(x) \geq 1/n$. For instance, we take $u_{0n}(x) = u_0(x) + 1/n$. Then by the maximum principle, $u_n(x, t) \geq 1/n$ is a classical $C^{2,1}$-smooth solution. It is convenient to fix a monotone sequence $\{u_{0n}\}$ decreasing in n; then the corresponding sequence of the solutions $\{u_n(x, t)\}$ is also monotonically decreasing (by the usual comparison) and hence the limit (2.94) always exists in the point-wise sense. For better convergence, extra estimates are needed which are similar to those given above. A continuous regularizing parameter $\varepsilon > 0$ can be used as well, where $u_{0\varepsilon}(x) = u_0(x) + \varepsilon$ gives a monotone in ε subset $\{u_\varepsilon(x, t) \geq \varepsilon\}$ of classical solutions. We refer to A.S. Kalashnikov's survey [202] and E. DiBenedetto's book [96] for further details.

Recall that the limit in (2.94) does not depend on the type of uniformly positive approximation of the data $\{u_{0n}\}$. Moreover, we may include a regular approximation of the equation (2.91) replacing it by the uniformly parabolic regularization

$$u_t = ((n^{-2} + u^2)^{m/2})_{xx} \quad \text{or} \quad u_t = (u^m + n^{-1}u)_{xx}.$$

In this case the regularization of continuous data u_0 is not necessary. [Recall the continuous approximation with $n^{-1} \mapsto \varepsilon > 0$.] Then we obtain the regularized sequence $\{u_n(x, t)\}$, and we need to pass to the limit $n \to \infty$ as above. In fact, this is quite a general principle in the theory of nonlinear singular PDEs, where "good" (we will use the term *proper*) solutions are only those which can be constructed, possibly in a unique way, by regular approximations of both the equation and initial-boundary data, i.e., via regularized problems. This is true for the PME, where in addition, weak solutions are the maximal ones, obtained by monotone decreasing approximations.

When dealing with nonlinear parabolic equations which exhibit singularity formation in finite time (blow-up or extinction), one of our preferred topics, the regular approximation technique is often the unique way to determine proper solutions existing beyond the singularity, and such an approach leads to an extended semigroup theory in the form of *discontinuous* limit semigroups. We refer for details to the paper [177], where the extended theory is developed by the authors. See further comments in the end section.

Continuing with the argument about the PME, we need to perform the same regularization for all the weak solutions involved in the Sturmian comparison, including the ZKB solutions. Thus, given an explicit solution $\mathcal{U} \in B$, we define the corresponding sequence

$$\mathcal{U} = \lim_{n\to\infty} \mathcal{U}_n,$$

by approximating the initial data $\{\mathcal{U}_{0n}\} \to \mathcal{U}(x,0)$ uniformly, with $\mathcal{U}_{0n}(x) \geq 1/n$. Note that we lose the explicit solution $\mathcal{U}$, but obtain a classical strictly positive approximation $\{\mathcal{U}_n\}$.

A crucial point of the approximation of both solutions is as follows: we perform the approximation of the initial data in such a way that the number of intersections does not depend on n, and moreover

$$I(0,\mathcal{U}_n) = I(0,\mathcal{U}) \quad \text{for any } n.$$

It is not difficult to find such an approximation. As usual, we deal with one or two and not more than three intersections, where the geometric configurations remain reasonably simple. Therefore, we can apply Sturm's results on sign changes $Z(t, w_n)$ of the difference $w_n = u_n - \mathcal{U}_n$ satisfying a linear parabolic equation with sufficiently smooth coefficients. Passing to the limit $n \to \infty$ yields a necessary estimate on the number of intersections $I(t,\mathcal{U}) = Z(t,w)$ of weak continuous solutions u and any $\mathcal{U} \in B$:

$$I(t,\mathcal{U}) \leq I(0,\mathcal{U}) \quad \text{for } t > 0.$$

Remark on dimensions. One must admit that the intersection comparison philosophy is essentially one-dimensional in space. No suitable notion of a nonincreasing-in-time "number" of intersections between solutions in $\mathbb{R}^N$ [as $(N-1)$-dimensional hypersurfaces] is available for any $N \geq 2$. (This question has a long history; let us cite a wrong "Herman theorem" in Courant–Hilbert's book [82], p. 454, and Arnol'd's survey [11] where multidimensional generalizations of Sturm theory are discussed with applications to geometric problems of curves and caustics.) Hence, we cannot treat in such a way the N-dimensional PME (2.2). But intersection comparison arguments apply to the radial solutions $u = u(r,t)$ with the single spatial variable $r = |x| \geq 0$, where the equation takes the form

$$u_t = \frac{1}{r^{N-1}}(r^{N-1}(u^m)_r)_r \quad \text{in } \mathbb{R}_+ \times \mathbb{R}_+,$$

with the symmetry condition at the origin

$$(u^m)(0,t) = 0 \quad \text{for } t > 0.$$

Therefore, a way of performing the asymptotic analysis is to establish sharp asymptotic properties of general radial solutions, and next use classical *symmetrization arguments* for the PME or related equations establishing that as $t \to \infty$, the solutions (possibly properly rescaled) become radially symmetric. This needs another kind of application of the maximum principle, the method of moving planes, which is a classical subject of the parabolic theory to be discussed below.

First example of intersection comparison for the PME. When dealing later on with PME-like equations, various (sometimes, rather involved) forms of intersection comparison with the ZKB-solutions will play an important part. Here we present a first simple application of intersection comparison establishing a sharp bound on the propagation of the interfaces of the weak solution $u(x,t)$ of the Cauchy problem (2.91). Let $s_u(t)$ denote the right-hand interface of the solution,

$$s_u(t) = \sup \operatorname{supp}(u(\cdot,t)), \quad t \geq 0.$$

By $M = \int u_0\,dx$ we denote the mass of initial data and hence of $u(\cdot,t)$ for any $t > 0$ by the mass conservation.

Proposition 2.19 *There exists a constant s_0 such that for any $t > 0$,*

$$s(t) \leq s_0 + s_{\mathcal{U}}(1+t) \equiv s_0 + c_m M^{(m-1)/(m+1)}(1+t)^{1/(m+1)}, \tag{2.95}$$

where $c_m > 0$ is a constant and $s_{\mathcal{U}}(t)$ is the right-hand interface of the explicit solution (2.23) *with the same mass M.*

Proof. We fix the ZKB-solution $\mathcal{U}(x - s_0, 1+t, C)$, where C is chosen from (2.40) so that

$$\int u(x,t)dx \equiv \int \mathcal{U}(x - s_0, 1+t, C)dx, \tag{2.96}$$

and s_0 is large enough to guarantee that the initial supports are disjoint:

$$\operatorname{supp}(u_0) \cap \operatorname{supp}(\mathcal{U}(x - s_0, 1, C)) = \emptyset \implies I(0,\mathcal{U}) = 1. \tag{2.97}$$

The shifting in time $t+1$ in $\mathcal{U}$ is performed for convenience to avoid comparison with the initial Dirac mass for $\mathcal{U}$ (though this can be done by approximation, i.e., taking the time variable $t + \varepsilon$ and setting $\varepsilon \to 0^+$). It follows from the intersection assumption in (2.97) that by the Sturm theorem

$$I(t,\mathcal{U}) \leq 1 \quad \text{for all } t > 0 \quad (\text{either} = 1 \text{ or} = 0). \tag{2.98}$$

Taking into account that all the interfaces are monotone in time (either increasing or decreasing), one can see that this immediately implies the comparison of the right-hand interfaces,

$$s_u(t) \leq s_0 + s_{\mathcal{U}}(1+t), \quad t > 0. \tag{2.99}$$

Indeed, in this simple geometric configuration, where by (2.96) both solutions have the same mass (the L^1-norm), one can see by drawing the spatial profiles of the solution that (2.99) cannot be violated for any $t > 0$ since this would mean the violation of the Sturmian property (2.98).

Actually, we have proved that in this case the number of intersections (known to be nonincreasing) is also nondecreasing, i.e., there holds

$$I(t, \mathcal{U}) \text{ does not decrease} \quad (\text{hence } \equiv 1).$$

The first half of this property is Sturm's theorem while the second half has nothing to do with Sturmian argument and, by geometry, is associated with the common property (2.96) of the solutions chosen. □

Choosing $s_0 \ll -1$, the same intersection comparison implies a similar lower estimate on the interface

$$s_u(t) \geq s_0 + s_{\mathcal{U}}(1+t), \quad t > 0,$$

which together with (2.95) give the precise asymptotic convergence of the interfaces:

$$t^{-1/(m+1)} s_u(t) \to c_m M^{(m-1)/(m+1)} \quad \text{as } t \to \infty.$$

Similar interface estimates can be proved by another version of the strong maximum principle, the *Shifting Comparison Principle*, we are going to describe next. Note that both principles, with some common features, have different areas of applications.

2.5.2 Shifting comparison principle (SCP)

This is a comparison result for one-dimensional equations that is, in essence, a form of the maximum principle for the equation after integration in x. It was presented in [300] and used in many later applications. Briefly stated, it says that for certain equations the following holds:

Shifting Comparison Principle. *Whenever comparison of the integrals of two initial functions holds, i.e.,*

$$\int_{-\infty}^{x} u_{01}(s)\,ds \geq \int_{-\infty}^{x} u_{02}(s)\,ds \quad \textit{for any } x \in \mathbb{R},$$

then the same comparison holds for the corresponding solutions at all times,

$$\int_{-\infty}^{x} u_1(s,t)\,ds \geq \int_{-\infty}^{x} u_2(s,t)\,ds \quad \text{for any } x \in \mathbb{R},\ t > 0.$$

Theorem 2.20 *The SCP holds for the nonnegative, weak and integrable solutions of the Cauchy problem for the filtration equation $u_t = \Phi(u)_{xx}$, where Φ is a monotone nondecreasing function, or more generally, a maximal monotone graph.*

The proof was done in [300] in the case of the PME, $\Phi(u) = u^m$, $m > 1$. It can be generalized rather easily to the p-Laplacian equation for $N = 1$, $u_t = \Delta_p(u)$ with $p > 1$, or even the more general equation with gradient dependence, $u_t = (\Phi(u_x))_x$.

The practical form of viewing the principle is this: shifting to the right a certain mass distribution, represented by a function $u_{01} \geq 0$, produces a distribution u_{02} where the above situation holds. Then, the same relation is preserved for all times.

Comparison of interfaces. This is one practical application that we will encounter in cases of finite propagation. Suppose we have two solutions with the same total mass

$$\int_{\mathbb{R}} u_1(x,t)\,dx = \int_{\mathbb{R}} u_2(x,t)\,dx = M > 0,$$

and let us assume that this quantity is conserved in time, as in the PME or PLE. Let us define the *right-hand interface* by the formula

$$s_i(t) = \max\{x : u_i(x,t) > 0\} \quad \text{for } t > 0,\ i = 1, 2.$$

Since this curve can also be described as

$$s_i(t) = \inf\{x : \int_{-\infty}^{x} u(s,t)\,ds = M\},$$

it is easily seen that a shifting comparison implies comparison of the interfaces

$$s_1(t) \leq s_2(t).$$

A similar argument by shifting can be applied to the left-hand interfaces with similar results.

Application to the PME. The SCP can be used in combination with the ZKB solutions to produce a very quick proof of the behaviour of the interfaces of the solutions of the PME.

Proposition 2.21 *Let u be a weak solution of the PME,* (2.91) *with nonnegative initial data $u_0 \in L^1(\mathbb{R})$ such that $u_0(x) = 0$ outside the bounded interval $I = (a,b)$. Let M be the mass of the solution and let $s_+(t)$ and $s_-(t)$ be the right-hand and left-hand interface respectively. Then $|s_\pm(t)| = O(t^{1/(m+1)})$ for $t \gg 1$. More precisely, we have*

$$\begin{cases} s_+(t) = c_m\, M^{(m-1)/(m+1)} t^{1/(m+1)} + O(1), \\ s_-(t) = -c_m\, M^{(m-1)/(m+1)} t^{1/(m+1)} + O(1), \end{cases} \tag{2.100}$$

as $t \to \infty$, where c_m is a positive constant that can be calculated explicitly from the formula of the ZKB solution.

The proof consists in shifting first all the mass to the left and concentrating it at the point $x = a$ in the form of a Dirac mass. The corresponding solution u_1 is the ZKB solution centered at $x = a$ with mass M, and its interfaces are exact

$$s_{1,+}(t) = c_m\, M^{(m-1)/(m+1)} t^{1/(m+1)} + a,$$
$$s_{1,-}(t) = -c_m\, M^{(m-1)/(m+1)} t^{1/(m+1)} + a.$$

The SCP says that they lie to the left of $s_+(t)$, $s_-(t)$ (i.e., the values lie below). The same argument applies to the concentration as a Dirac delta to the right, at $x = b$ and we obtain

$$s_+(t) \le s_{2,+}(t) = c_m\, M^{(m-1)/(m+1)} t^{1/(m+1)} + b,$$
$$s_-(t) \le s_{2,-}(t) = -c_m\, M^{(m-1)/(m+1)} t^{1/(m+1)} + b.$$

This proves the result. Let us remark that a further (and delicate) argument is used in the paper [307] to conclude the finer result

$$\begin{cases} s_+(t) = c_m\, M^{(m-1)/(m+1)} t^{1/(m+1)} + b + o(1), \\ s_-(t) = -c_m\, M^{(m-1)/(m+1)} t^{1/(m+1)} + a + o(1). \end{cases}$$

With this the convergence of the solution towards the ZKB profile easily follows in one space dimension.

2.5.3 Other comparisons

Symmetrization. The way to apply the two preceding comparison arguments in the general context of several space dimensions has not been found. A partial answer is the concept of spherical rearrangement of functions and symmetrization, which has a large mathematics literature. Let us take a simple case: given a function $f \in L^1(\mathbb{R}^N)$, $f \ge 0$, we define its radially symmetric rearrangement as the radially symmetric function $f_*(r)$, $r = |x|$, that has level sets of the same measure, i.e.,

$$\text{meas}\{x : f(x) > k\} = \text{meas}\{x : f_*(|x|) > k\}.$$

With the condition of right- or left-continuity, this defines f_* in a unique way.

We then define the relation of *mass concentration* for radially symmetric solutions as follows:

Definition. *f is less concentrated than g, in symbols $f \prec g$, iff for every $r > 0$,*

$$\int_{B_r(0)} f(x)\,dx \le \int_{B_r(0)} g(x)\,dx. \tag{2.101}$$

The following result is proved in [298].

Theorem 2.22 *Let u_i, $i = 1, 2$, be a pair of radially symmetric solutions of the problem for the PME with initial data u_{0i} in the class of nonnegative and integrable data. Assume that $u_{01} \prec u_{02}$. Then, for every $t > 0$,*

$$u_1(\cdot, t) \prec u_2(\cdot, t). \tag{2.102}$$

But we can also compare a general function f with a radially symmetric function g. Then, $f \prec g$ *iff for every* $r > 0$,

$$\int_{\Omega} f(x)\,dx \le \int_{B_r(0)} g(x)\,dx \tag{2.103}$$

for every set Ω *with the same or less measure than the ball* $B_r(0)$. In that case the result stays valid.

As an immediate consequence of this result we obtain comparison of supports.

Proposition 2.23 *Let* u_1 *and* u_2 *be two solutions as before and let us assume that they are compactly supported and have the same mass. If* $R_1(t)$ *and* $R_2(t)$ *are the radii of their respective supports, we have*

$$R_1(t) \le R_2(t) \qquad \text{for every } t \ge 0. \tag{2.104}$$

We only need to observe that if the common mass is M, then

$$R_i(t) = \sup\{r > 0 : \int_{B_r(0)} u_i(x,t)\,dx < M\}.$$

Moving plane method. Let us mention another comparison principle to be used in our future analysis. It is *Aleksandrov's Reflection Principle* or *method of moving planes* [1], [2]. It plays a fundamental role in the theory of nonlinear parabolic and elliptic equations.

Let us show a simple and quite useful application in the form of a *monotonicity lemma*:

Proposition 2.24 *Let* $u \ge 0$ *be a solution of the Cauchy problem for the heat equation, the PME or the PLE with initial data supported in the ball* $B_R(0)$, $R > 0$. *Then for every* $x_0 \in \mathbb{R}^N$ *such that* $|x_0| > R$ *and every* $t > 0$, $u(x,t)$ *is monotone nonincreasing along the ray* $l(x_0) = \{x = s\,x_0 : s \ge 1\}$ *in the sense that*

$$u(s_2x_0, t) \le u(s_1x_0, t) \quad \text{if } s_2 \ge s_1 \ge 1. \tag{2.105}$$

Proof. The application of Aleksandrov's reflection principle proceeds as follows: we draw the hyperplane H which is mediatrix between the points $x = s_2x_0$ and $y = s_1x_0$ in the above situation. It is easy to see that H divides the space $\mathbb{R}^N$ into two half-spaces, one Ω_1, which contains y and the support of u_0, and another one, Ω_2, which contains x and where $u_0 = 0$. We consider now the initial and boundary-value problem in $\widehat{Q} = \Omega_1 \times (0,\infty)$. Two particular solutions of this problem are compared: one of them is u_1, the restriction of u to $\widehat{Q}$, another one is

$$u_2(z,t) = u(\pi(z),t), \quad z \in \Omega_1,$$

where π is the specular symmetry with respect to the hyperplane H. This is where the equation appears for the first time: it has to be invariant under this symmetry. Thus, if we orient the coordinate axes so that $H = \{x_1 = 0\}$, then

$$\pi(x_1, ..., x_n) = (-x_1, ..., x_n).$$

Clearly, u_1 and u_2 are solutions of the same equation in $\widehat{Q}$. Besides, $u_1 = u_2$ on the lateral boundary, $\Sigma = H \times (0, \infty)$. Finally, $u_1 \geq u_2$ for $t = 0$ since $u_2 = 0$ in Ω_1. Since our equations satisfy the maximum principle, by comparison for the mixed problem we have

$$u_1(z, t) \geq u_2(z, t) \quad \text{for } z \in \Omega_1,\ t > 0.$$

Putting $z = y$ we have $\pi(z) = x$ so that $u(y, t) \geq u(x, t)$ as desired. □

Actually, the result can be sharpened into monotonicity along the cone of directions with vertex x_0, with axis along the direction of x_0 and with a certain amplitude (angle) that allows for the previous argument with hyperplanes to be applied.

Let us show another typical application to the large time distribution of the level sets of solutions of the PME in the form of a *monotonicity lemma* as is used in [71].

Proposition 2.25 *Let $u \geq 0$ be a solution of the Cauchy problem for the PME with initial data supported in the ball $B_R(0)$, $R > 0$. Then for every x such that $|x| > 2R$ and every $r < |x| - 2R$, $r > 0$, we have*

$$u(x, t) \leq \inf_{|y|=r} u(y, t). \tag{2.106}$$

Proof. We use Aleksandrov's reflection principle as before. Now we draw the hyperplane H which is mediatrix between the points x and y in the above situation. It is easy to see that H divides the space $\mathbb{R}^N$ into two half-spaces; we define them to be Ω_1 and Ω_2 as before, perform the specular transformation π, apply comparison, and finally conclude that $u(y, t) \geq u(x, t)$ as desired. □

It follows from here that the lower level lines tend to be almost spherical. In particular, this applies to the free boundary, cf. [210]. The argument applies without changes to the heat equation, the p-Laplacian equation and other parabolic equations as long as they respect specular symmetry and the maximum principle.

Remarks and comments on the literature

§ 2.1. There are many texts that develop the general aspects of reaction-diffusion equations, like Smoller's [293]. General existence, uniqueness, comparison and regularity results for quasilinear degenerate parabolic equations with arbitrary diffusion nonlinearities or other singular coefficients can be found in Kalashnikov's survey [202], see also Di Benedetto's book [96] and general regularity results in [94], [95]. The theory of the filtration equation was mainly developed by Bénilan, Crandall and coworkers. Unfortunately, their text on the subject is still unpublished; cf. [41]. Main impetus for the regularity comes from the work of Caffarelli and coworkers [68, 66].

§ 2.2. The mathematical study of the PME can be said to begin with the investigation on the existence of the source-type solutions, cf. the original papers [325],

[25], [261], which introduce the need for generalized solutions. The first study of existence and uniqueness is the famous paper by O.A. Oleinik, A.S. Kalashnikov and Czhou Yui-Lin' [257] for the PME and filtration equations with general nonlinearities. Since the early 1970s there has been a continuous stream of new results, starting with the works of Aronson on regularity and the interface (free boundary) [14], Bénilan and coworkers on general well-posedness and semigroups [34], and Kamin on asymptotic behaviour [203]. The shift in the functional setting from L^2 spaces towards L^1 and measures is a favorite theme in Brezis's work, [53, 55]. The study of the regularity of solutions and free boundaries was largely developed by Aronson, Caffarelli, Friedman and coworkers. Existence under optimal conditions is due to the combined efforts of Aronson, Caffarelli, Bénilan, Crandall, Pierre, Dalhberg and Kenig, among others. General presentations of the results can be found in the survey papers [15] and [264] and the text [304]. A theory of viscosity solutions for the PME is developed in [70].

Maximum principles, super- and subsolutions are well known in the classical parabolic and elliptic theory, see the books [118], [272], [293], and admit formulations in all sorts of generalized solutions which have a great role in the theory. For nonlinear comparison with weak super- and subsolutions cf. [202].

The free boundary problems have been extensively researched in recent times. In particular, the Stefan problem is maybe the most important free boundary problem of evolution type [283, 249]. The combustion free boundary problem for the heat equation admits a free boundary on which we impose two conditions

$$u = 0, \quad |\nabla u| = c,$$

where $c > 0$ is a constant or a given function, cf. the survey paper [306]. It is also called the Florin problem, [114].

§ 2.3. A main reference for the techniques of scaling and similarity applied to porous media and other nonlinear equations are Barenblatt's books, [26, 27]. The subsection on the Cauchy problem is taken from [307], where the literature is discussed. The main theorem of convergence for $m > 1$ is essentially due to Friedman and Kamin [123], but the whole proof is in [307], which also contains the convergence in relative error for $m < 1$. The result for the Dirichlet problem is due to Aronson and Peletier [19], and the outline of proof is taken from [308]. The reader will find a survey of results for the Cauchy–Neumann problem in [3], for the exterior Dirichlet problem in [275]. Refined asymptotic expansion for the PME is quite involved, see an application of perturbation theory of linear operators in [7] (cf. earlier formal analysis in [5, 323]). Strong asymptotic properties of geometrical type (asymptotic concavity) have been recently obtained, cf. [237]; they apply in all dimensions and can be generalized to the p-Laplacian equation [238].

§ 2.4. For Lyapunov's original work we refer to [244]. Concerning parabolic equations, we mention results by T.I. Zelenyak [327] who showed that a standard (integral) Lyapunov function can be constructed for any quasilinear uniformly parabolic equation $u_t = a(x, u, u_x)u_{xx} + b(x, u, u_x)$ on a bounded interval with general nonlinear boundary conditions (including a delicate result on the ω-limit sets of any bounded orbit). Further general extensions are due to H. Matano [247], who showed that Sturm's theorem on zeros generates a *discrete* Lyapunov function (the number of intersections with any stationary solution; Matano used the term "lap-number" [248]). This geometric Lyapunov approach applies to general fully nonlinear parabolic equations $u_t = F(x, u, u_x, u_{xx})$.

The first Lyapunov approach is rather folklore after the work of Bénilan on contraction semigroups [34]; it has been developed in detail in [307]. Newman's Lyapunov approach was proposed in [252] and developed in [276]. It is base of current work on so-called entropy methods for which we refer to [73]–[75], where other references can be found. This method allows us to obtain rates of convergence that improve the result of Theorem 2.3.

§ 2.5. The first Sturm theorem was formulated as a consequence of (ii) in the section; it is a form of the strong maximum principle for parabolic equations. As a by-product of the first theorem, Sturm presented an evolution proof of bounds of the number of zeros of eigenfunction expansions: for finite Fourier series

$$f(x) = \sum_{L \le k \le M} (a_k \cos kx + b_k \sin kx), \quad x \in [0, 2\pi], \tag{2.107}$$

by using the PDE (2.88) with $q \equiv 0$ (plus periodic boundary conditions), it was proved that $f(x)$ has at least $2L$ and at most $2M$ zeros. (Sturm also presented an ODE proof of the same result to be compared with Liouville's proof who also was interested in this ODE subject.) This lower bound is often referred as to the *Hurwitz Theorem*, which was wider known than the first Sturm PDE theorem. This *Sturm–Hurwitz theorem* is the origin of many striking results, ideas and conjectures in topology of curves and symplectic geometry. We recommend the book by W.T. Reid [277], entirely devoted to generalizations and applications of Sturm's ideas and theorems to the ODE theory, as well as V.I. Arnol'd's surveys [12], [11] on related questions of symplectic geometry. These references contain detailed descriptions of the results, historical comments and extensive lists of earlier references.

Unlike the classical Sturm theorems on zeros of ODEs, Sturm's evolution zero set analysis for parabolic PDEs did not attract much attention in the nineteenth century and, in fact, it was practically forgotten for almost a century. It seems that G. Pólya (1933) [270] was the first to revive the interest in the twentieth century for the first Sturm theorem applied to the heat equation (the number of "Nullstellen" of $u(x, t)$, the number of $x \in [0, 2\pi]$ such that $u(x, t) = 0$, was studied via Sturm's approach; radial and cylindrical solutions were considered and zero properties of convolution integrals were also described).

The earlier extension by A. Hurwitz (1903) [198] of Sturm's result on zeros of (2.107) to infinite Fourier series (2.107) with $M = \infty$ did not use PDEs. Since the 1930s, versions of the Sturmian argument were rediscovered on several occasions. For instance, a key idea of the Lyapunov monotonicity analysis in the famous KPP-problem by A.N. Kolmogorov, I.G. Petrovskii and N.S. Piskunov (1937) [223] on the stability of travelling waves in reaction-diffusion equations was based on the first Sturm theorem in a simple geometric configuration. We have mentioned Nickel's [253] and Sattinger's paper [287] in the 1960s.

From the 1980s the Sturmian argument for PDEs began to penetrate more and more into the theory of linear and nonlinear parabolic equations, and found several fundamental applications. These are asymptotic stability theory for various nonlinear parabolic equations, orbital connections and transversality of stable-unstable manifolds for semilinear parabolic equations as Morse–Smale systems (we refer to the pioneer D. Henry's paper [192] and [6]), unique continuation theory, Floquet bundles and a Poincaré–Bendixson theorem for parabolic equations, problems of symplectic geometry and curve shortening flows, etc. An extended list of references from different areas can be found in Section 4 of [138] and in the books [141] and [286], Chapter 4. In subsequent chapters, intersections comparison via Sturm's theorem will be used in proving important estimates on solutions of nonlinear parabolic equations, where we refer again to the results which are most important for our applications.

The construction of solutions by approximation is a main issue in nonlinear studies involving limited regularity. Maybe the most classical area of such a construction in the twentieth century happens in the theory of shock waves or the theory of conservation laws (vanishing viscosity method), where we mention pioneer work of O.A. Oleinik and S.N. Kruzhkov [256], [227] in constructing entropy solutions in the 1950–60s, the general theory of viscosity solutions of Hamilton–Jacobi equations in 1970s and 80s by P.L. Lions, M.G. Crandall and coworkers [83]–[86], and various other impressive generalizations and extensions to wide classes of nonlinear PDEs.

The shifting comparison principle was introduced in [300] for the study of the PME. Similar results for more general quasilinear heat equations can be proved by intersection comparison; see [128] and [286], p. 245. A careful checking shows that the arguments by shifting comparison and intersection comparison are not completely equivalent even for the PME.

A general reference to symmetrization, also called Schwarz symmetrization, is [190]. The application to obtaining suitable a priori estimates for elliptic problems is described by Weinberger in [320], 1962. The technique has been described in detail by Bandle in [24], 1980, which covers a wide number of elliptic and also parabolic problems. See also [296, 217, 250]. The introduction of the relation of mass concentration for one-dimensional or radial solutions seems to be due to Hardy. The general definition is used in [298] and then extended to the p-Laplacian in [299]. These concepts have been used and extended by a number of authors like Diaz [92]. A review of our work will appear in [309].

Pioneering applications of Aleksandrov's reflection principle are due to Serrin [290]. It plays a big role in the theory of nonlinear parabolic and elliptic equations, see Chapt. 9 in [183]. A famous application of symmetrization phenomena for nonlinear elliptic and parabolic problems is described by Gidas, Ni and Nirenberg in [182]. Another symmetrization argument, based on Aleksandrov's reflection principle, is given in Section 5 of [210].

3

Equation of Superslow Diffusion

In this chapter we present a first application of the abstract stability theory developed in Chapter 1. We start with a simple model, involving a single nonlinear operator, namely, a nonlinear version of the heat equation, of the form $u_t = \Delta\Phi(u)$*, where* $\Phi(u)$ *is an increasing real-valued function.*

The asymptotic behaviour of the solutions of the heat equation is well known in various settings and under different boundary conditions. We have recalled this fact in the previous chapter along with the results for the power case $\Phi(u) = u^m$ with $m > 1$ (the PME), where the asymptotic self-similar behaviour was established by scaling and Lyapunov techniques, i.e., standard techniques as we call them.

We consider here the case of the exponential nonlinearity

$$\Phi(u) = e^{-E/u}, \qquad u > 0, \tag{3.1}$$

which is known to play an important role in heat conductivity, combustion and in general thermodynamics as the famous *Arrhenius law* occurring in many of the coefficients describing thermodynamical properties of nonlinear media. $E > 0$ is a constant (an energy) that we will normalize to 1 in the sequel. The asymptotic behaviour of this equation produces a case of asymptotically small perturbations that fits perfectly the theory developed in the first chapter. We study two settings: the boundary value problem in a bounded space domain, and the Cauchy problem in the whole line, $x \in \mathbb{R}$.

3.1 Asymptotics in a bounded domain

Main result. Let Ω be a bounded domain with smooth boundary $\partial\Omega$. We study the asymptotic behaviour of the solution to the initial and boundary-value problem

$$u_t = \Delta\Phi(u) \quad \text{in } Q = \Omega \times (0, \infty) \quad \text{with } \Phi(u) = e^{-1/u}, \tag{3.2}$$

$$u(x, 0) = u_0(x) \quad \text{for } x \in \Omega, \tag{3.3}$$

$$u(x,t) = 0 \quad \text{on } \partial\Omega \times [0,\infty), \tag{3.4}$$

where the initial function $u_0 \not\equiv 0$ satisfies

$$u_0 \in L^\infty(\Omega), \quad u_0 \geq 0 \quad \text{in } \Omega. \tag{3.5}$$

Equation (3.2) is an example of an *equation of superslow diffusion*, so called because the heat-conduction coefficient $\Phi'(u) = u^{-2}e^{-1/u}$ is very small as $u \to 0$, more precisely $\Phi'(u)$ grows more slowly than any power of u for $u \approx 0$. Existence and uniqueness of a nonnegative weak solution and comparison theorems for (3.2)–(3.5) with quite general monotone increasing functions Φ are well known, see comments at the end.

In order to present our asymptotic result in the dynamical systems framework we have developed in Chapter 1, we need some preliminary transformations. Consider a solution $u = u(x,t)$ of (3.2)–(3.5). First, we introduce the natural variable

$$v(x,t) = e^{-1/u(x,t)}. \tag{3.6}$$

Therefore, $0 \leq v(x,t) < 1$. It satisfies the quasilinear equation

$$v_t = v\,(\ln v)^2\,\Delta v \quad \text{in } Q \tag{3.7}$$

(which does not look better on first inspection), zero boundary conditions and the initial conditions corresponding to applying (3.6) to u_0. Let $F_\Omega(x) \equiv F(x;\Omega)$ be the unique classical, positive solution in Ω of the linear inhomogeneous elliptic problem

$$\Delta F = -1 \quad \text{in } \Omega, \qquad F = 0 \quad \text{on } \partial\Omega. \tag{3.8}$$

This is our main result.

Theorem 3.1 *With the above assumptions, uniformly in $x \in \Omega$,*

$$\lim_{t\to\infty} t\,(\ln t)^2\, v(x,t) = F_\Omega(x). \tag{3.9}$$

The result departs from the asymptotic behaviour of the solutions to the mixed problem (3.2)–(3.4) for simpler Φ's like the heat equation ($\Phi = u$) and the PME with $\Phi = u^m$, cf. Section 2.3. Before proceeding with the proof, we are going to elaborate a bit on the meaning of the result.

APPROXIMATE SEPARATION OF VARIABLES FOR v. Theorem 3.1 shows that the equation of superslow diffusion (3.2) has the property of separated-variables asymptotics in terms of the natural variable v, although there exist *no* nontrivial solutions of problem (3.2)–(3.4) of the separated-variables form. In fact, the function

$$V(x,t) = (t\ln^2 t)^{-1}\, F(x;\Omega), \tag{3.10}$$

which represents the common asymptotic behaviour of the solutions to our problem, is only an *approximate self-similar solution* to (3.7), increasingly accurate as t grows to infinity. Indeed, if we assume that for large t, there is an approximate separation of variables, $v \sim h(t)F(x)$, with $h(t) \to 0$ as $t \to \infty$, a heuristic computation gives for all $t \gg 1$,

$$h'(t)F \sim h^2(t)\,\ln^2 h(t)\; F\Delta F,$$

hence $h(t) \sim (t\,\ln^2 t)^{-1}$. Function $V(x,t)$ in (3.10), which describes the asymptotic behaviour of our class of solutions to equation (3.7) in the bounded domain Ω, satisfies the nonautonomous quasilinear parabolic equation

$$v_t = (\ln^2 t + 2\ln t)\, v\,\Delta v\,, \tag{3.11}$$

which upon the change of variables $ds/dt = (\ln t)^2 + 2\ln t$ becomes

$$v_s = v\,\Delta v\,. \tag{3.12}$$

Though equations (3.7) and (3.12) look quite different, our result amounts to saying that the separated-variables solution of (3.12), $\overline{V}(s,t) = s^{-1}F_\Omega(x)$, explains the asymptotic behaviour of a wide class of solutions of equation (3.2).

THE MESA PATTERN FOR u. We are able to observe the phenomenon of variable separation by working with the function v. If we translate our result (3.9) to the variable u thanks to

$$u(x,t) = -1/\ln v(x,t)\,, \tag{3.13}$$

we obtain

$$\lim_{t\to\infty} (\ln t)\, u(x,t) = 1 \tag{3.14}$$

locally uniformly in Ω, whence a flat profile. Notice that the information about the spatial structure (the *spatial pattern*) is lost and in these variables we only see a *mesa-like* profile. The limit cannot be uniform in Ω because of the boundary condition $u = 0$. Finally, the function $U(x,t) = -1/\ln V(x,t)$, which is an approximate self-similar solution to (3.2), is an exact solution of the equation

$$u_t = (\ln^2 t + 2\ln t)\, u^2 \Delta e^{-1/u}\,, \tag{3.15}$$

and the same comments made above for v apply now to (3.2) and (3.15).

Rescaled equation. The proof of the theorem is based on the study of the rescaled variable θ naturally corresponding to our asymptotic behaviour, which is defined as

$$\theta(x,\tau) = (T+t)\ln^2(T+t)\, v(x,t), \tag{3.16}$$

where $\tau = \ln(T+t)$ and $T > 1$ is a large fixed constant. Using this trick is not a matter of chance. On the contrary, such a rescaling is a quite useful tool in similar problems that we will see often in the sequel. We would like to draw the attention of the reader to the proper choice of the time factor, which is crucial and not always easy to predict.

We shall consider the initial- and boundary-value problem for $\theta(x, \tau)$ consisting of the equation

$$\begin{aligned}\theta_\tau = \mathbf{B}(\theta, \tau) \equiv \mathbf{A}(\theta) + \frac{4 \ln \tau}{\tau}\theta\, \Delta\theta + \frac{2}{\tau}\,(\theta - \theta \ln \theta\, \Delta\theta) \\ + \frac{4 \ln^2 \tau}{\tau^2}\theta\, \Delta\theta - \frac{4 \ln \tau}{\tau^2}\theta \ln\theta\, \Delta\theta + \frac{1}{\tau^2}\theta\,(\ln\theta)^2\, \Delta\theta, \quad (3.17)\end{aligned}$$

in $\Omega \times (\tau_0, \infty)$, $\tau_0 = \ln T > 0$, with initial data

$$\theta(x, \tau_0) = \theta_0(x) \equiv T \ln^2 T\, e^{-1/u_0(x)} \quad \text{in } \Omega, \qquad (3.18)$$

and Dirichlet boundary condition

$$\theta(x, \tau) = 0 \quad \text{on } \partial\Omega \times [\tau_0, \infty)\,. \qquad (3.19)$$

The autonomous part of equation (3.17) has the form

$$\mathbf{A}(\theta) = \theta\, \Delta\theta + \theta\,. \qquad (3.20)$$

In Theorem 3.1 we prove the convergence of $\theta(x, \tau)$ as $\tau \to \infty$ to the unique solution $F(x; \Omega)$ of the stationary problem

$$\mathbf{A}(F) = 0 \quad \text{in } \Omega\,, \qquad F > 0 \quad \text{in } \Omega, \qquad F = 0 \quad \text{on } \partial\Omega, \qquad (3.21)$$

which is equivalent to (3.8).

Exact upper and lower estimates. Here we prove upper and lower estimates for $v(x, t)$ with exact decay rates. They are based on the construction of suitable super- and subsolutions. We begin in the first lemma with the supersolution. Given $R > 0$, we denote by $F_R(x)$ the function

$$F_R(x) = \frac{R^2 - |x|^2}{2N} > 0 \quad \text{in } B_R = \{|x| < R\}\,, \qquad (3.22)$$

which solves (3.8) in $\Omega = B_R$. We have

Lemma 3.2 *Let $R > 0$ be such that $B_R \supset \overline{\Omega} \equiv$ closure(Ω) and let $c > 1$. There exists $\tau_1 > 1$ such that the function*

$$\overline{\theta}(x, \tau) = c\, F_R(x) \qquad (3.23)$$

is a classical strictly positive supersolution of (3.17) *in* $\Omega \times (\tau_1, \infty)$.

Proof. Choosing R such that $\overline{\Omega} \subset B_R$, we have $\overline{\theta} > 0$ in $\overline{\Omega} \times (\tau_1, \infty)$, and hence the proof consists in checking that this function satisfies the corresponding parabolic differential inequality

$$\mathbf{B}(\overline{\theta}, \tau) \le \overline{\theta} \equiv 0 \quad \text{in } \overline{\Omega} \times (\tau_1, \infty)\,. \qquad (3.24)$$

Since $\Delta F_R = -1$ in $\overline{\Omega}$, one can calculate that for $\tau > \tau_1 > 1$,

$$\mathbf{B}(\overline{\theta}, \tau) = cF_R\Big\{1 - c - \frac{4\ln\tau}{\tau}c + \frac{2}{\tau}[1 + c\ln(cF_R)] \\ - \frac{4\ln^2\tau}{\tau^2}c + \frac{4\ln\tau}{\tau^2}c\ln(cF_R) - \frac{1}{\tau^2}c\ln^2(cF_R)\Big\}.$$

Hence, we get

$$\mathbf{B}(\overline{\theta}, \tau) \le cF_R\Big\{1 - c + \frac{2}{\tau}[1 + c\,|\ln(c\|F_R\|_\infty)|] \\ + \frac{4\ln\tau}{\tau^2}c\,|\ln(c\|F_R\|_\infty)|\Big\}. \quad (3.25)$$

It is easy to see that if $\tau_1 = \tau_1(c, R)$ is large enough, the right-hand side of (3.25) is negative for $\tau > \tau_1$ and hence (3.24) is valid. □

In order to construct a suitable subsolution, for arbitrary small fixed $\lambda > 0$, we consider the function $f(x)$ determined as follows: f is nonnegative, radially symmetric and satisfies in its positivity set the degenerate elliptic equation

$$f(\ln f)^2\Delta f + f = 0 \quad (3.26)$$

with $\|f\|_\infty = \lambda$. Therefore, if $r = |x|$, $f = f(r;\lambda)$ will satisfy

$$\frac{1}{r^{N-1}}(r^{N-1}f')' + (\ln f)^{-2} = 0 \quad \text{in } \mathbb{R}_+ \cap \{f > 0\}, \quad (3.27)$$

$$f(0) = \lambda, \quad f'(0) = 0 \quad (3.28)$$

($f' = df/dr$). One can see that there exists a unique classical solution of (3.27), (3.28), which is positive and smooth in some interval $[0, r_0)$, and it vanishes at the endpoint $r = r_0(\lambda) > 0$. We have the estimate

$$r_0(\lambda) \le \sqrt{2N\int_0^\lambda (\ln z)^2 dz}. \quad (3.29)$$

Notice that $f'(r_0) < 0$. It will be convenient to extend f to $r > r_0$ by 0. Finally, note that by (3.28) and (3.29)

$$\|f\|_\infty \to 0, \quad \operatorname{diam}(\operatorname{supp}(f)) \to 0 \quad \text{as } \lambda \to 0. \quad (3.30)$$

Lemma 3.3 *Fix an arbitrary* $x_0 \in \Omega$ *and a large* $\tau_2 > 1$. *Then there exists* $\lambda_0 \in (0, 1)$ *such that for any* $\lambda \in (0, \lambda_0]$ *the function*

$$\underline{\theta}(x, \tau) = f(|x - x_0|; \lambda) \quad (3.31)$$

is a weak subsolution of (3.17) *in* $\Omega \times (\tau_2, \infty)$ *if* $\{x : |x - x_0| < r_0(\lambda)\} \subset \Omega$.

Proof. Firstly, we note that by (3.30) $\operatorname{supp}(f) \subset \Omega$ for small $0 < \lambda < 1$. Secondly, we shall now check that

$$\mathbf{B}(f, \tau) \geq 0 \quad \text{in } \{|x - x_0| < r_0(\lambda)\} \times (\tau_2, \infty). \tag{3.32}$$

Since $f\Delta f = -f/(\ln f)^2$ in $\operatorname{supp}(f)$, one can calculate that

$$\mathbf{B}(f, \tau) = f\left\{1 - \frac{1}{\ln^2 f} - \frac{4 \ln \tau}{\tau \ln^2 f} + \frac{2}{\tau}\left(1 + \frac{1}{\ln f}\right) - \frac{4 \ln^2 \tau}{\tau^2 \ln^2 f} + \frac{4 \ln \tau}{\tau^2 \ln f} - \frac{1}{\tau^2}\right\}$$

and hence, using the condition $\|f\|_\infty = \lambda$, we get the estimate in $S = \operatorname{supp}(f) \times (\tau_2, \infty)$,

$$\mathbf{B}(f, \tau) \geq f\left\{1 - \frac{1}{\ln^2 \lambda} - \frac{4 \ln \tau_2}{\tau_2 \ln^2 \lambda} - \frac{2}{\tau_2 |\ln \lambda|} - \frac{4 \ln^2 \tau_2}{\tau_2^2 \ln^2 \lambda} - \frac{4 \ln \tau_2}{\tau_2^2 |\ln \lambda|} - \frac{1}{\tau_2^2}\right\} \geq 0$$

if τ_2 is large enough and $\lambda \in (0, 1)$ is sufficiently small. It is now clear that $\underline{\theta}(x, \tau)$ satisfies the inequality $\theta_\tau \leq \mathbf{B}(\theta, \tau)$ in the sense of distributions in $\Omega \times (\tau_2, \infty)$. Observe that $\underline{\theta}(x, \tau)$ is stationary, while actual solutions of the equation have expanding supports. □

We are now ready to obtain lower and upper bounds for the solution $\theta(x, \tau)$ of (3.17)–(3.19) for large τ. We begin with an upper estimate.

Lemma 3.4 *There exist $C > 0$ and $R > 0$ such that for $\tau_3 \gg 1$,*

$$\theta(x, \tau) \leq C\, F_R(x) \quad \text{in } \Omega \times (\tau_3, \infty). \tag{3.33}$$

Proof. Fix some large constants $c > 1$, $R > 0$ and $\tau_1 \geq \tau_0$ as in Lemma 3.2 such that (3.23) is a classical positive supersolution of (3.17) in $\Omega \times (\tau_1, \infty)$. Since $\|u(\cdot, t)\|_\infty \to 0$ as $t \to \infty$, an obvious result for such parabolic equations in divergent form, we can find a large $T_* > T_1 = e^{\tau_1}$ such that

$$cF_R(x) \geq T_1 (\ln T_1)^2 \exp\{-1/u(x, T_*)\} \tag{3.34}$$

in Ω. We now introduce another rescaled variable

$$\tilde{\theta}(x, \tilde{\tau}) = (T_1 + t - T_*) \ln^2(T_1 + t - T_*)\, v(x, t), \tag{3.35}$$

with the new time variable $\tilde{\tau} = \ln(T_1 + t - T_*)$. Remark that $t = T_*$ corresponds to $\tilde{\tau} = \tau_1$. One can see that $\tilde{\theta}(x, \tilde{\tau})$ satisfies the same equation (3.17) with τ replaced by $\tilde{\tau}$ and hence by Lemma 3.2, under the above assumptions, function (3.23) is a classical supersolution for $\tilde{\tau} \geq \tau_1$. Since (3.34) means that $\tilde{\theta}(x, \tau_1) \leq cF_R(x)$ in Ω, and $\tilde{\theta}(x, \tilde{\tau}) = 0 \leq F_R(x)$ on the lateral boundary $x \in \partial\Omega$, $\tilde{\tau} \geq \tau_1$, we conclude by the maximum principle that

$$\tilde{\theta}(x, \tilde{\tau}) \leq c\, F_R(x) \quad \text{in } \Omega \times (\tau_1, \infty). \tag{3.36}$$

But from (3.16) and (3.35) we have $\tilde{\theta}(x, \tilde{\tau}) = \theta(x, \tau)(1+o(1))$ as $\tau \to \infty$ uniformly in Ω. Hence there exists some constant $C \geq c$ such that upper estimate (3.33) holds in $\Omega \times (\tau_3, \infty)$ if τ_3 is large enough. □

We now establish a lower estimate.

Lemma 3.5 *There exists a function $f_*(x) > 0$ in Ω, $f_* = 0$ on $\partial\Omega$, such that for any large τ_4,*

$$\theta(x, \tau) \geq f_*(x) \quad in \;\; \Omega \times (\tau_4, \infty)\,. \tag{3.37}$$

Proof. By well-known properties of weak solutions to (3.2), the support expands without bounds, so that there exists $T_2 > 1$ such that $u(x, t) > 0$ in Ω for $t \geq T_2$. Let $\tau_4 = \max\{\tau_2, \ln(T + T_2)\}$, where τ_2 is given in Lemma 3.3. Given $x_0 \in \Omega$ we consider those parameters $\lambda \in (0, \lambda_0)$ such that the support of $f(|x - x_0|; \lambda)$ is contained in Ω, and moreover we have

$$\theta(x, \tau_4) \geq f(|x - x_0|; \lambda) \quad \text{in } \Omega\,. \tag{3.38}$$

This means that we have to choose $\lambda \in (0, \lambda_1(x_0))$. Thanks to Lemma 3.3, we can conclude that the same inequality will hold for $\tau \geq \tau_4$. Hence, if we define

$$f_*(x) = \sup\{f(x - x_0; \lambda) : x_0 \in \Omega,\ \lambda \in (0, \lambda_1(x_0)),\} \tag{3.39}$$

then estimate (3.37) holds. □

The lower estimate (3.37) means that $\|\theta(\cdot, \tau)\|_\infty \geq C_* = \|f_*\|_\infty$ for large τ, where the constant $C_* > 0$ may depend on the initial function. It is equivalent to the following lower estimate of the solution $v(x, t)$ of (3.7):

$$\|v(\cdot, t)\|_\infty \geq C_*(t \ln^2 t)^{-1}(1 + o(1)) \quad \text{as } t \to \infty\,, \tag{3.40}$$

which for $u(x, t)$ satisfying (3.2) reads: as $t \to \infty$,

$$\|u(\cdot, t)\|_\infty \geq [\ln(C_*^{-1}(t \ln^2 t)]^{-1}(1 + o(1)) = (\ln t)^{-1}(1 + o(1)). \tag{3.41}$$

Proof of Theorem 3.1. We want to prove that $\theta(x, \tau)$ converges towards $F(x; \Omega)$ as $\tau \to \infty$ uniformly in $x \in \Omega$. We shall use the S-Theorem from Chapter 1, and view equation (3.17) for θ, as a perturbation of the evolution equation

$$\theta_\tau = \mathbf{A}(\theta) = \theta\, \Delta\theta + \theta\,. \tag{3.42}$$

As a functional space, where the orbit lies, we take, in view of (3.33) and (3.37),

$$X = \{g \in L^\infty(\Omega) :\ f_*(\cdot) \leq g(\cdot) \leq C\, F_R(\cdot) \text{ a.e. in } \Omega\}, \tag{3.43}$$

which forces us to consider only large enough times, to be precise $\tau \geq \tau_* = \max_i\{\tau_i\}$, where τ_i, $i = 1, \ldots, 4$ are as in the above four lemmas. We want to apply the S-Theorem, which says that any orbit of the perturbed, possibly nonautonomous equation (in this case (3.17)) converges towards the ω-limit set of the asymptotic equation (here (3.42)) if the three hypotheses (H1)–(H3) are satisfied.

Let Ω_* be the ω-limit set of equation (3.42) in X. It is well known that **A** generates a continuous semigroup in the class of bounded *strictly positive* initial data. One can see that under the condition $\theta(\cdot,\tau) \in X$ for $\tau \geq \tau_*$ (and hence $\theta(x,\tau) \geq f_*(x) > 0$ in Ω for all $\tau \geq \tau_*$) the ω-limit set of any solution $u \in C([0,\infty) : X)$ of (3.42) consists of the unique stationary solution of the equation in X which is positive in Ω, namely

$$\Omega_* = \{F_\Omega(\cdot)\}. \tag{3.44}$$

Indeed, equation (3.42) admits the *Lyapunov function*

$$\phi(\theta(\tau)) = \tfrac{1}{2}\|D_x\theta(\tau)\|_2^2 - \|\theta(\tau)\|_1\,, \tag{3.45}$$

which is nonincreasing for $\tau > \tau_*$, so that we have for any $s > 0$,

$$\phi(\theta(\tau+s)) - \phi(\theta(\tau)) \leq -\int_\tau^{\tau+s}\int_\Omega \frac{(\theta_\tau(z))^2}{\theta(z)}\,dx\,dz \leq 0\,.$$

This corresponds to the formal calculation

$$\frac{d}{d\tau}\phi(\theta(\tau)) = -\int_\Omega \frac{(\theta_\tau(\tau))^2}{\theta(\tau)}\,dx \leq 0\,.$$

By standard asymptotic results this implies (3.44), the global asymptotic stability of the unique positive stationary solution.

Thus, in order to prove that the ω-limit of the solution $\theta(\cdot,\tau)$ of (3.17) with initial value at $\tau = \tau_*$ in the space X defined by

$$\omega(\theta(\cdot,\tau_0)) = \{f(\cdot) \in X : \exists \text{ a sequence } \{\tau_j\} \to \infty \text{ such that } \theta(\cdot,\tau_j) \to f(\cdot)\}$$

is contained in $\Omega_* = \{F_\Omega\}$ and the orbit $\{\theta(\cdot\,,\tau), \tau > \tau_*\}$ approaches Ω_* uniformly as $\tau \to \infty$, we must check whether the hypotheses (H1)–(H3) are satisfied.

From Lemmas 3.4 and 3.5 we have that all solutions of (3.17) with values $\theta(\cdot,\tau_*) \in X$, stay in X for $\tau > \tau_*$. Moreover, standard regularity theory implies then that the orbits $\{\theta(\tau)\}$ are relatively compact subsets in the space $L^\infty_{\rm loc}((\tau_*,\infty) : X)$. This is the first hypothesis, (H1), of the S-Theorem.

Hypothesis (H2) translates the fact that (3.17) is an asymptotically small perturbation of (3.42) and consists in showing that solutions of (3.17) converge asymptotically to solutions of (3.42) in the following sense: if $\theta(\tau)$ is a solution of (3.17) and $\{s_j\}$ is a divergent sequence of positive times such that $\theta(s_j+\tau)$ converges in $L^\infty_{\rm loc}((\tau_*,\infty) : X)$ to a function $\Theta(\tau)$, then $\Theta(\tau)$ is a solution of (3.42). We remark that this condition is very weak and is straightforward.

Finally, hypothesis (H3) demands that the ω-limit set of equation (3.42) in X be nonvoid, compact and uniformly stable in the sense of Lyapunov. Since such a set has one element, $F(x;\Omega)$, we only need to establish the uniform stability, which comes from the following result.

Lemma 3.6 *For every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that if $\theta(\tau)$ is any solution of* (3.42) *which is positive in Ω and*

$$\|\theta(0) - F_\Omega\|_\infty \leq \delta\,, \tag{3.46}$$

then

$$\|\theta(\tau) - F_\Omega\|_\infty \leq \varepsilon \quad \textit{for any} \quad \tau > 0. \tag{3.47}$$

Proof. Fix an arbitrarily small $\nu > 0$ and let

$$\Omega_\nu^+ = \Omega \cup \{x \in \mathbb{R}^N \backslash \Omega : \rho(x, \partial\Omega) \equiv \inf_{y \in \partial\Omega} |x - y| < \nu\}, \tag{3.48}$$

$$\Omega_\nu^- = \Omega \backslash \{x \in \Omega : \rho(x, \partial\Omega) < \nu\}. \tag{3.49}$$

Obviously, the sets $\Omega_\nu^\pm$ are bounded domains in $\mathbb{R}^N$ with smooth boundaries $\partial\Omega_\nu^\pm$, $\Omega_\nu^- \subset \Omega \subset \Omega_\nu^+$, and

$$\rho(\partial\Omega_\nu^+, \partial\Omega_\nu^-) \equiv \inf_{x \in \partial\Omega_\nu^+,\, y \in \partial\Omega_\nu^-} |x - y| \to 0 \quad \text{as } \nu \to 0.$$

We introduce the functions $F(x; \Omega_\nu^\pm)$ defined as in (3.8). For convenience, we extend the functions $F(x; \Omega_\nu^\pm)$ outside of $\Omega_\nu^\pm$ by 0. By the standard comparison

$$F(x; \Omega) < F(x; \Omega_\nu^+) \quad \text{in } \overline{\Omega}, \qquad F(x; \Omega_\nu^-) < F(x; \Omega) \quad \text{in } \overline{\Omega_\nu^-}. \tag{3.50}$$

Using the monotonicity of the families $F(x; \Omega_\nu^+)$ and $F(x; \Omega_\nu^-)$ in ν, it is easy to show that as $\nu \to 0$,

$$F(x; \Omega_\nu^\pm) - F(x; \Omega) \to 0 \quad \text{uniformly in } \Omega. \tag{3.51}$$

Now, given $\varepsilon > 0$, we choose $\nu > 0$ such that

$$|F(x; \Omega_\nu^\pm) - F(x; \Omega)| \leq \varepsilon\,. \tag{3.52}$$

By the strong maximum principle, there exists $\delta > 0$ such that

$$F(x; \Omega_\nu^+) \geq F(x; \Omega) + \delta \quad \text{in } \Omega, \qquad F(x; \Omega) \geq F(x; \Omega_\nu^-) + \delta \quad \text{in } \Omega_\nu^-\,. \tag{3.53}$$

In view of this, if $\theta \in X$ and $\|\theta(0) - F(x; \Omega)\|_\infty \leq \delta$, we will have

$$F(x; \Omega_\nu^-) \leq \theta(0) \leq F(x; \Omega_\nu^+) \quad \text{in } \Omega\,. \tag{3.54}$$

We now use the fact that $F(x; \Omega_\nu^+)$ is a classical solution of (3.42) in $Q = \Omega \times (0, \infty)$ with $F(x, \Omega_\nu^+) > 0 = \theta(x, \tau)$ on the lateral boundary, $x \in \partial\Omega$, $\tau \geq 0$, to conclude by the maximum principle that

$$\theta(x, \tau) \leq F(x; \Omega_\nu^+) \quad \text{in } Q\,.$$

A similar comparison performed in $\Omega_\nu^- \subset \Omega$ between $F(x; \Omega_\nu^-)$ and $\theta(x, \tau)$ gives

$$F(x; \Omega_\nu^-) \leq \theta(x, \tau) \quad \text{in } Q\,.$$

Taking into account (3.52), we finally get

$$F(x; \Omega) - \varepsilon \leq \theta(x, \tau) \leq F(x; \Omega) + \varepsilon \quad \text{in } Q\,,$$

which proves our result. □

3.2 The Cauchy problem in one dimension

Main results, comparison and discussion. We now investigate the asymptotic behaviour of the solution to the Cauchy problem for the equation of superslow diffusion in one dimension

$$u_t = (e^{-1/u})_{xx} \quad \text{in } Q = \mathbb{R} \times (0, \infty), \tag{3.55}$$

with initial condition

$$u(x, 0) = u_0(x) \quad \text{in } \mathbb{R}. \tag{3.56}$$

We assume that u_0 satisfies

$$u_0 \geq 0, \quad u_0 \not\equiv 0 \quad \text{in } \mathbb{R}, \quad u_0 \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R}). \tag{3.57}$$

Existence and uniqueness of a continuous nonnegative weak solution of (3.55)–(3.56) is well known; see comments at the end of the chapter. The solution is smooth at any point of positivity. As in the case of bounded domains, we state our main result in terms of the function

$$v = e^{-1/u}. \tag{3.58}$$

Then, $0 \leq v(x,t) < 1$ in Q, and $v(x,t)$ solves the quasilinear equation

$$v_t = v\,(\ln v)^2\, v_{xx} \quad \text{in } Q. \tag{3.59}$$

The asymptotic behaviour of $v(x,t)$ is exactly described by the following result.

Theorem 3.7 *Under hypotheses* (3.57) *we have*

$$\lim_{t\to\infty} t\, v(\eta(\ln t), t) = F_a(\eta) \equiv \tfrac{1}{2}(a^2 - \eta^2)_+ \tag{3.60}$$

uniformly for $\eta \in \mathbb{R}$, *where* a *is one-half of the initial mass:*

$$a = \tfrac{1}{2} \int u_0(x)\, dx > 0. \tag{3.61}$$

If we translate this result (3.60) to the function $u(x,t)$ by means of the inverse transformation

$$u(x,t) = -1/\ln v(x,t), \tag{3.62}$$

we get the asymptotic formula

$$\lim_{t\to\infty} (\ln t)\, u(\eta(\ln t), t) = 1 \tag{3.63}$$

uniformly in any set $\{|\eta| \leq c\}$, where $c \in (0, a)$ is a constant, while for $|\eta| \geq a$ we have

$$\lim_{t\to\infty} (\ln t)\, u(\eta(\ln t), t) = 0. \tag{3.64}$$

Thus, in terms of the initial variable $u(x, t)$ we observe a *mesa-like profile*. Notice that the only parameter which appears in the formulas is the normalized length of the support of u, namely $2a$. This parameter is easily calculated from the law of conservation of mass $\int u(x, t)dx$ = constant, since for large t it follows from (3.63), (3.64) that $\int u(x, t)dx \approx 2a$ and $\|u_0\|_1 = 2a$. Any further information about the asymptotic spatial structure of the solution as $t \to \infty$ is lost in the u variable (in first approximation).

It is interesting to compare Theorem 3.7 with the asymptotic behaviour of the solution to the initial-boundary value problem for equation (3.55) in a bounded domain studied in the previous section. Let us recall the result obtained there for the one-dimensional case where the domain Ω is a finite interval $(-l, l)$, $l > 0$, and

$$u(x, 0) = u_0(x) \quad \text{in } (-l, l), \quad u = 0 \quad \text{for } x = \pm l;\ t > 0, \tag{3.65}$$

where $u_0 \in L^\infty$, $u_0 \geq 0$, $u_0 \not\equiv 0$. Then, uniformly in $(-l, l)$,

$$\lim_{t\to\infty} t(\ln t)^2 v(x, t) \to F_l(x) \equiv \tfrac{1}{2}(l^2 - x^2). \tag{3.66}$$

Two differences appear. Firstly, the rate of decay is $(t \ln^2 t)^{-1}$ as compared with t^{-1} in (3.60). Secondly, the particular asymptotic profile is determined by the length of domain $2l$ and not by the initial function (in a bounded domain, the boundary information is dominant for large times over the initial data).

Going back to the Cauchy problem (3.55), (3.56), we also obtain a precise result on the asymptotic behaviour of interfaces of every compactly supported solution.

Theorem 3.8 *Assume that* (3.57) *holds and also that* u_0 *has a compact support. Then as* $t \to \infty$,

$$s_+(t) \equiv \sup\{x \in \mathbb{R} : u(x, t) > 0\} = a \ln t + O(1),$$

$$s_-(t) \equiv \inf\{x \in \mathbb{R} : u(x, t) > 0\} = -a \ln t + O(1).$$

Let us make some comments before proceeding with the proofs. In order to understand the appearance of the asymptotic profile $F_a(\eta)$, it is convenient to view our result in terms of the rescaled function θ corresponding to our asymptotic formula (3.60), which is defined by

$$\theta(\eta, \tau) = (2 + t)\, v(\eta \ln(2 + t), t) \tag{3.67}$$

(the number 2 plays no special role, any number $T > 0$ would do). Then $\theta(\eta, \tau)$ solves the Cauchy problem

$$\theta_\tau = \mathbf{B}(\theta, \tau) \equiv \mathbf{A}(\theta) + \frac{1}{\tau}[\theta_\eta \eta - 2\theta(\ln\theta)\theta_{\eta\eta}] + \frac{1}{\tau^2}\theta(\ln\theta)^2\theta_{\eta\eta} \tag{3.68}$$

in $\mathbb{R} \times (\tau_0, \infty)$, with initial condition

$$\theta(\eta, \tau_0) = \theta_0(\eta) \equiv 2\exp\{-1/u_0(\eta \ln 2)\}. \tag{3.69}$$

The autonomous part of the operator in the right-hand side of (3.68) has the form

$$\mathbf{A}(\theta) = \theta\theta_{\eta\eta} + \theta. \tag{3.70}$$

It is easily seen that the functions $F_a(\eta)$ given in (3.60) are precisely the radially symmetric nonnegative weak solutions of the stationary equation $\mathbf{A}(\theta) = 0$ which are monotone nonincreasing in $|\eta|$. Therefore, Theorem 3.7 amounts to proving the convergence of the solution $\theta(\eta, \tau)$ as $\tau \to \infty$ to the corresponding stationary solution

$$\mathbf{A}(F) = 0 \quad \text{in } \mathbb{R}, \quad F \geq 0, \quad F = F(|\eta|), \tag{3.71}$$

which is uniquely determined by the total mass of the initial function, see (3.61). Moreover, the function

$$V(x, t) = t^{-1} F_a(x/\ln t), \tag{3.72}$$

describing by (3.60) the asymptotic behaviour of the solution $v(x, t)$ as $t \to \infty$, satisfies the nonautonomous quasilinear parabolic equation

$$v_t = (\ln t)^2 v v_{xx} - \frac{x}{t \ln t} v_x, \tag{3.73}$$

which looks quite different from (3.59). Thus, (3.72) is an approximate self-similar solution of equation (3.59).

As for equation (3.55), the function $U(x, t) = -1/\ln V(x, t)$ (its approximate self-similar solution) is in fact an explicit self-similar solution of the quasilinear equation

$$u_t = (\ln t)^2 u^2 (e^{-1/u})_{xx} - \frac{x}{t \ln t} u_x. \tag{3.74}$$

Preliminaries. Explicit solutions. A weak solution to the problem (3.55), (3.56) is a continuous nonnegative function which is smooth at any point where $u > 0$ and has a continuous heat flux $-(e^{-1/u})_x$ on interfaces $\{u = 0\}$; see comments at the end of the chapter. We also note that for the solution of nonlinear equations of the type (3.55), the law of conservation of mass holds, i.e., if the initial mass is finite

$$\int u_0(x)\, dx = E_0 > 0, \tag{3.75}$$

then

$$\int u(x, t)\, dx = E_0 \quad \text{for any } t > 0. \tag{3.76}$$

In view of (3.58), this implies that

$$-\int_{-\infty}^{\infty} dx/\ln v(x,t) = -\int_{-\infty}^{\infty} dx/\ln v_0(x) = E_0 \quad \text{for } t \geq 0. \tag{3.77}$$

The proof of our result is also based on a careful use of a family of explicit solutions. It turns out that by a nonlocal Lie–Bäcklund transformation the solutions of the superslow diffusion equation are transformed into solutions of the quasilinear heat equation

$$u_t = (k(u)u_x)_x \tag{3.78}$$

with the exponential nonlinearity $k(u) = e^u$. It is known [65] that two equations (3.78) with coefficients $k(u)$ and $K(u) = u^{-2}k(u^{-1})$ are equivalent (a kind of homology driven by a Lie–Bäcklund group of nonlocal transformations). Setting $k(u) = e^u$ yields $K(u) = u^{-2}e^{-1/u}$ whence the equation of superslow diffusion. Unlike equation (3.78) with general nonlinearity $k(u)$, the exponential equation with $k(u) = e^u$ admits extra symmetries and exact self-similar solutions. This makes it possible to translate one of them into the following explicit solution of (3.59) (see comments):

$$v_*(x,t;c) = \frac{1}{2t}(c^2 - w^2)_+, \tag{3.79}$$

where $c > 0$ is a fixed arbitrary constant and the function $w = w(x,t;c) \in [0,c)$ is determined from the algebraic equation

$$|x| = \Phi(w,c) \equiv [2 + \ln(2t)]w + (c-w)\ln(c-w) - (c+w)\ln(c+w). \tag{3.80}$$

Since the function $\Phi(w,c)$ in the right-hand side satisfies

$$\Phi'_w = \ln(2t) - \ln(c^2 - w^2) \geq 0$$

for fixed $c > 0$ and $t \geq c^2/2$, equation (3.80) uniquely determines the function $w(x,t;c) \in [0,c)$ in terms of $x \in [0, x_*(t;c))$, where

$$x_*(t;c) = c\ln t + c\ln\left(e^2/2c^2\right) = c\ln t(1+o(1)) \quad \text{as } t \to \infty. \tag{3.81}$$

Then (3.79) is an even, continuous and nonnegative function defined for $x \in \mathbb{R}$, $t \geq c^2/2$ and satisfying

$$v_*(x,t;c) = 0 \text{ for } |x| \geq x_*(t;c), \quad v_*(x,t;c) > 0 \text{ for } |x| < x_*(t;c), \tag{3.82}$$

$$\sup_{x\in\mathbb{R}} v_*(x,t;c) = v_*(0,t;c) = \frac{c^2}{2t} < 1 \quad \text{for } t > \frac{c^2}{2}. \tag{3.83}$$

Going back to the variable u by means of (3.62), we get the explicit compactly supported solution

$$u_*(x,t;c) = -1/\ln v_*(x,t;c) \tag{3.84}$$

of equation (3.55). Indeed, one can calculate from (3.79), (3.80) that

$$\exp\left\{-\frac{1}{u_*(x,t;c)}\right\} = \frac{1}{t}\left\{\frac{x_*(t)-|x|}{|\ln(x_*(t)-|x|)|}\right\}(1+o(1))$$

near the interfaces $x = \pm x_*(t; c)$, and hence $\exp\{-1/u_*\} \in C^1$, which implies the continuity of the heat flux on the interfaces. The total mass of the explicit solution is preserved,

$$\|u_*(\cdot,t;c)\|_{L^1(\mathbb{R})} = 2c \quad \text{for } t \geq c^2/2. \tag{3.85}$$

It is curious that at $t_0 = c^2/2$ the function $u_*(x, t_0; c)$ behaves near $x = 0$ like $|x|^{-2/3}$, which of course is an integrable singularity, but not a δ-function as for the ZKB solutions of the PME.

We begin with some simple properties of these explicit solutions.

Lemma 3.9 *For any fixed $c > 0$, uniformly in $\eta \in \mathbb{R}$,*

$$v_*(x,t;c) = \frac{1}{t}F_c(\eta) + O\left(\frac{1}{t\ \ln t}\right) \quad \text{as } t \to \infty. \tag{3.86}$$

Proof. Using (3.80), we obtain

$$|\eta| = w + \frac{\ln(2e^2t/(t+2))}{\ln(t+2)}w + \frac{(c-w)\ln(c-w)-(c+w)\ln(c+w)}{\ln(t+2)}$$

for $w \in (0, c)$. Hence, $w(x, t; c) = |\eta| + O(1/\ln t)$ as $t \to \infty$ in $\{|x| \leq x_*(t; c)\}$, which by (3.79) completes the proof. □

Lemma 3.10 *For any fixed $0 < c_1 < c_2$, there holds*

$$v_*(x,t;c_2) > v_*(x,t;c_1) \quad \text{in } \{|x| < x_*(t;c_2)\},\ t > 2c_2^2. \tag{3.87}$$

Proof. First, we note that

$$\frac{d}{dc}x_*(t;c) > 0 \quad \text{for } t > 2c^2. \tag{3.88}$$

Using (3.79), we get

$$\frac{d}{dc}v_*(x,t;c) = \frac{1}{t}(c - w\,w_c'), \tag{3.89}$$

and (3.80) yields that $w_c'(x, t; c)$ is well defined in $\{|x| < x_*(t; c)\}$ for $t > c^2/2$. One can see that

$$w_c' = \frac{\ln(c+w)-\ln(c-w)}{\ln(2t)-\ln(c-w)-\ln(c+w)} < 1 \tag{3.90}$$

for $w \in (0, c)$, $t > 2c^2$. This together with (3.89) implies that $c - w\,w_c' > 0$ for $w \in (0, c)$, and hence by (3.89)

$$\frac{d}{dc} v_*(x, t; c) > 0 \quad \text{in } \{|x| < x_*(t; c)\}, \quad t > 2c^2. \tag{3.91}$$

Using (3.88), (3.91), we get (3.87) completing the proof. □

First estimates. Let $u(x, t)$ be the solution of the problem (3.55), (3.56). Assume that u_0 has a compact support in an interval $[-b, b]$. We begin with an upper estimate of this solution.

Lemma 3.11 *There exist constants $c_1 > 0$ and $t_1 > c_1^2/2$ such that*

$$v(x, t) \le v_*(x, t_1 + t; c_1) \quad \textit{in } \mathbb{R} \times \mathbb{R}_+. \tag{3.92}$$

Proof. By the comparison theorem, we obtain that (3.92) will be valid if

$$v(x, 0) \le v_*(x, t_1; c_1) \quad \text{in } \mathbb{R}. \tag{3.93}$$

Using properties (3.81)–(3.83), we have that (3.93) holds if

$$t_1 > c_1^2/2, \quad \sup_{x\in\mathbb{R}} v_*(x, t_1; c_1) = c_1^2/2t_1 > \sup_{x\in\mathbb{R}} v(x, 0) = M_1 \in (0, 1), \tag{3.94}$$

$$x_*(t_1; c_1) = c_1 \ln\left(e^2 t_1/2c_1^2\right) \gg l_1 = \sup\{|x| : x \in \operatorname{supp} v(x, 0)\}. \tag{3.95}$$

Choose t_1 as follows: $t_1 = (c_1^2/4)(1 + 1/M_1)$. Then (3.94) hold and (3.95) implies

$$c_1 \ln\left[\frac{e^2}{8}\left(1 + \frac{1}{M_1}\right)\right] \gg l_1,$$

which is valid for any $c_1 > 0$ large enough. □

Our next estimate is a lower bound.

Lemma 3.12 *There exist constants $c_2 > 0$ and $t_2 > c_2^2/2$ such that*

$$v(x, t) \ge v_*(x, t; c_2) \quad \textit{in } \mathbb{R} \times (t_2, \infty). \tag{3.96}$$

Proof. The support of any nontrivial solution expands without bounds as $t \to \infty$, so that there exists $t_2 \ge 0$ such that $u(0, t_2) > 0$ and $u(x, t_2) \in C(\mathbb{R})$. Choose arbitrarily small $c_2 > 0$. Then from (3.81)–(3.83) one can see that inequality $v(x, t_2) \ge v_*(x, t_2; c_2)$ in $\mathbb{R}$ holds, and hence by the comparison theorem, estimate (3.96) is valid. □

If we now perform the change of variables (3.67), then from Lemmas 3.9, 3.11 and 3.12 and properties (3.81)–(3.83) of the explicit solutions, we get the following weak form of the asymptotic behaviour, which in particular determines the rate of stabilization to 0 of $u(x, t)$.

Lemma 3.13 *If* u_0 *satisfies* (3.57) *and has a compact support, then there exist* $\tau_* > 0$ *and constants* $0 < c_- < c_+$ *such that*

$$F(\eta; c_-) \le \theta(\eta, \tau) \le F(\eta; c_+) \quad in \ \mathbb{R} \times (\tau_*, \infty). \tag{3.97}$$

As a consequence of these estimates, we can also control the growth of the support of the solution $u(x, t)$ as $t \to \infty$.

Corollary 3.14 *There exist* $t_* > 0$ *and* $0 < C_- < C_+$ *such that for* $t \ge, t_*$

$$\left\{|x| < C_- \ln\left(e^2 t/2C_-^2\right)\right\} \subseteq \operatorname{supp} u(\cdot, t) \subseteq \left\{|x| < C_+ \ln\left(e^2 t/2C_+^2\right)\right\}, \tag{3.98}$$

$$-\left[\ln\left(C_-^2/2t\right)\right]^{-1} \le \sup_{x\in\mathbb{R}} u(x, t) \le -\left[\ln\left(C_+^2/2t\right)\right]^{-1}. \tag{3.99}$$

A sharp estimate. We establish here a sharp lower bound.

Lemma 3.15 *There holds*

$$\liminf_{t\to\infty} t \sup_{x\in\mathbb{R}} v(x, t) \ge \tfrac{1}{2}a^2, \tag{3.100}$$

where a is the half mass given by (3.61).

Proof. Step 1. Assume also that u_0 has a compact support. By the indefinite expansion property of the support of the solution to the Cauchy problem (3.55), (3.56), there exists $t = t_1$ such that the support

$$\operatorname{supp} v(x, t_1) = (l_-, l_+) \tag{3.101}$$

is a connected interval and $0 \in (l_-, l_+)$. By Aleksandrov's reflection principle (see Section 2.5 and comments at the end of the chapter), the solution $v(x, t)$ is a monotone function with respect to x in $(-\infty, -b) \cup (b, \infty)$ for any fixed $t \ge t_1$.

Step 2. Fix now an arbitrarily small $\varepsilon > 0$. We replace $v(x, t_1)$ by an approximation $\tilde{v}_\varepsilon(x)$ such that

(i) $\tilde{v}_\varepsilon(x) \le v(x, t_1)$ in $\mathbb{R}$ and $\tilde{v}_\varepsilon(x) \equiv v(x, t_1)$ in $(l_- + \varepsilon, l_+ - \varepsilon)$,
(ii) $\int (u(x, t_1) - \tilde{u}_\varepsilon(x))dx \le 2\varepsilon$, and
(iii) $\left|\frac{d}{dx}\tilde{v}_\varepsilon(x)\right| \ge 1$ near the endpoints of its support.

Construction. Consider the behaviour close to the right-hand interface, $x \approx l_+$. It is clear that we can choose $l_1 \in (l_+ - \varepsilon/2, l_+)$ such that

$$\int_{l_1}^{l_+} u(x, t_1)dx < \tfrac{1}{2}\varepsilon.$$

To the left of l_1 we draw the line $y(x) = M(l_1 - x)$. This line intersects the graph of $v(x, t_1)$ for the first time in a point $l_2 < l_1$. If $M > 1$ is large enough, we have $l_2 > l_+ - \varepsilon$ and

$$\int_{l_2}^{l_+} u(x, t_1)dx < \varepsilon.$$

For such an M we define $\tilde{v}_\varepsilon(x) = v(x, t_1)$ if $0 \le x \le l_2$, $\tilde{v}_\varepsilon(x) = y(x)$ if $l_2 \le x \le l_1$ and $\tilde{v}_\varepsilon(x) = 0$ if $x \ge l_1$. The same construction holds for the left-hand side $x < 0$.

Step 3. Denote by $v_\varepsilon(x, t)$ the weak solution of the Cauchy problem in $\mathbb{R} \times (t_1, \infty)$ for the equation (3.59) with the initial function $v_\varepsilon(x, t_1) = \tilde{v}_\varepsilon(x)$ in $\mathbb{R}$. Let

$$c_\varepsilon = \tfrac{1}{2} \int \tilde{u}_\varepsilon(x)dx \equiv \tfrac{1}{2} \int u_\varepsilon(x, t)dx \quad \text{for every } t \ge t_1,$$

so that $a - \varepsilon \le c_\varepsilon < a$. Since by construction $\tilde{v}_\varepsilon(x) \le v(x, t_1)$ in $\mathbb{R}$, from the comparison theorem we have $v_\varepsilon(x, t) \le v(x, t)$ in $\mathbb{R} \times (t_1, \infty)$.

We now consider the family of explicit solutions $\{v_*(x - x_0, t + T; c_\varepsilon), x_0 \in [-b, b], T > 0\}$ having the same mass c_ε as $u_\varepsilon(x, t)$. For a fixed $t \ge t_1$, we denote by $I(t; x_0, T)$ the number of sign changes in $\mathbb{R}$ of the difference $w(x, t; x_0, T) \equiv v_\varepsilon(x, t) - v_*(x - x_0, t + T; c_\varepsilon)$ or, which is the same, the *intersection number* in $\mathbb{R}$ of the functions $v_\varepsilon(x, t)$ and $v_*(x - x_0, t + T; c_\varepsilon)$. By the intersection comparison (see Section 2.5), we have that $I(t; x_0, T)$ does not increase with time and, in particular,

$$I(t; x_0, T) \le I(t_1; x_0, T) \quad \text{for } t > t_1. \tag{3.102}$$

Notice that by known C^∞-regularity (and analyticity) of the weak solution at positivity points, we may conclude that for $t \ge t_1$ every zero of the difference in the positivity domain of both solutions considered is an isolated point. Since by the properties of the explicit solutions given above, we have for an arbitrary fixed $x_0 \in [-b, b]$,

$$v_*(x - x_0, t_1 + T; c_\varepsilon) \approx c_\varepsilon^2/2(t_1 + T) \quad \text{as } T \to \infty$$

uniformly in x on compact subsets of $\mathbb{R}$, by using the property (iii) of the function $v_\varepsilon(x, t_1)$, we have that for every $x_0 \in [-b, b]$ and T large enough,

$$I(t_1; x_0, T) = 2. \tag{3.103}$$

This together with (3.102) yields the inequality

$$I(t; x_0, T) \le 2 \quad \text{for } t \ge t_1. \tag{3.104}$$

Fix an arbitrary $x_0 \in [-b, b]$ and $T = T_0$ large enough. We now prove that for $t > t_1$,

$$\sup_{x \in \mathbb{R}} v_\varepsilon(x, t) \ge \sup_{x \in \mathbb{R}} v_*(x - x_0, t + T_0; c_\varepsilon). \tag{3.105}$$

Assume for a moment that this is true. Then

$$\sup_{x \in \mathbb{R}} v_*(x - x_0, t + T_0; c_\varepsilon) \equiv v_*(0, t + T_0; c_\varepsilon) = c_\varepsilon^2/2(t + T_0),$$

and (3.105) implies that

$$\liminf_{t\to\infty} t \sup_{x\in\mathbb{R}} v(x,t) \ge \liminf_{t\to\infty} t \sup_{x\in\mathbb{R}} v_\varepsilon(x,t) \ge \tfrac{1}{2}c_\varepsilon^2.$$

Since $\varepsilon > 0$ is arbitrary, we obtain the desired result (3.100). Let us prove (3.105).

Step 4. Suppose (3.105) is not valid and

$$t_* = \sup\{\tau_1 + \tau > 0,\ (3.105) \text{ holds for all } t \in [t_1, t_1 + \tau]\} < \infty.$$

Let $x_* \in [-b, b]$ be a point of maximum of the function $v_\varepsilon(x, t_*)$ and hence by a definition of t_*, we have

$$v_\varepsilon(x_*, t_*) = c_\varepsilon^2/2(t_* + T_0). \tag{3.106}$$

Consider the explicit solution $v_*(x - x_*, t + T_0; c_\varepsilon)$. By construction,

$$w(x, t_*; x_*, T_0) \equiv v_\varepsilon(x, t_*) - v_*(x - x_*, t_* + T_0; c_\varepsilon) = 0 \quad \text{for } x = x_*, \tag{3.107}$$

and $w_x(x, t_*; x_*, T_0) = 0$ for $x = x_*$.

Suppose first that $x = x_*$ is a *tangency point* of the functions $v_\varepsilon(x, t_*)$ and $v_*(x - x_*, t_* + T_0; c_\varepsilon)$, i.e., the difference $w(x, t_*; x_*, T_0)$ satisfying (3.107) does not change sign in a small neighbourhood of the point $x = x_*$. Since these have the same masses,

$$I(t_*; x_*, T_0) \ge 1. \tag{3.108}$$

Indeed, if (3.108) is not valid and $I(t_*; x_*, T_0) = 0$, then by the strong maximum principle it follows that, since $v_\varepsilon \not\equiv v_*$, for arbitrarily small $\delta > 0$ either $v_\varepsilon(x, t_* + \delta) < v_*(x - x_*, t_* + T_0 + \delta; c_\varepsilon)$ or $v_\varepsilon(x, t_* + \delta) > v_*(x - x_*, t_* + T_0 + \delta; c_\varepsilon)$ in the domain of positivity of both functions, contradicting the equality of masses. Hence, there exists at least one point of intersection, i.e., a point x_1 where the difference w changes sign, and $x_1 \ne x_*$. Assume without loss of generality that the difference $w(x, t_*; x_*, T_0) \le 0$ in a small neighbourhood $J_r = (x_* - r, x_* + r)$ of the point $x = x_*$ with $r \ll |x_1 - x_*|$. Then by using the continuous dependence of the function $v_*(x - x_*, t_* + T_0; c_\varepsilon)$ with respect to a small perturbation of the value of T_0, we obtain that for any small $\delta > 0$, there exist at least two points of sign change for the perturbed difference $w(x, t_*; x_*, T_0 + \delta)$ in J_r, one to the left of $x = x_*$ and one to the right, and also an intersection point which lies not far from x_1, and anyway is outside J_r. Therefore, for small $\delta > 0$, we have

$$I(t_*; x_*, T_0 + \delta) \ge 3. \tag{3.109}$$

This leads to a contradiction with (3.104) for $t = t_*$, $x_0 = x_* \in [-b, b]$ and $T = T_0 + \delta$.

Now, if the maximum $x = x_*$ is an *inflection point* for the difference $w(x, t_*; x_*, T)$ satisfying (3.107), namely that it changes sign in any neighbourhood of the point $x = x_*$, then we easily show that such inflection can occur from, at least, three points of intersection. Assume without loss of generality that $w(x, t_*; x_*, T_0) > 0$

in a small left-hand neighbourhood of $x = x_*$ and $w(x, t_*; x_*, T_0) < 0$ in a small right-hand one. Then it is easily seen that for any $\lambda > 0$ small enough there holds

$$I(t_*; x_* - \lambda, T_0) \geq 3 \tag{3.110}$$

holds, contradicting (3.104) for $t = t_*$, $x_0 = x_* - \lambda$ and $T = T_0$ and completing the proof of Lemma 3.15 in the case of compactly supported data.

Step 5. If u_0 is not compactly supported, the proof is made by approximation from below with compactly supported functions. □

Semiconvexity. For the proof of Theorem 3.7, we need also the following lower estimate of the second derivative of the solution.

Lemma 3.16 *Let $u(x,t)$ be a solution of* (3.55), (3.56). *Then for every $\varepsilon > 0$, there exists $T_\varepsilon > 0$ such that in the domain $\{(x,t) : x \in \mathbb{R}, t \geq T_\varepsilon\}$*

$$v_{xx} \geq -\frac{(1+\varepsilon)}{t \ln^2 t}. \tag{3.111}$$

Proof. We derive a *semiconvexity* estimate; see comments at the end of the chapter. By approximation, we may assume that u_0 is continuous, bounded and positive in $\mathbb{R}$. Then $v(x,t)$ is a classical solution of equation (3.59). Differentiating it twice with respect to x, we obtain the equation satisfied by $z = v_{xx}$:

$$\begin{aligned} z_t = v(\ln v)^2 z_{xx} &+ 2[(\ln v)^2 + 2\ln v]v_x z_x \\ &+ \frac{2}{v}(\ln v + 1)(v_x)^2 z + [(\ln v)^2 + 2\ln v]z^2. \end{aligned} \tag{3.112}$$

We try an explicit subsolution for this equation in the form $z(t) = -1/\varphi(t)$, with $\varphi > 0$. Then we easily check that a sufficient condition is that

$$\varphi'(t) \leq \inf_{x\in\mathbb{R}} \left[(\ln v(x,t))^2 + 2\ln v(x,t)\right]. \tag{3.113}$$

Now, for large $t > 0$, from Lemma 3.13 we have $v \leq \text{const}/t$, hence $\ln^2 v + 2\ln v \geq (1 - \varepsilon/4)\ln^2 t$ for ε small if t is large enough. Therefore, an admissible choice is

$$\varphi(t) = (1 - \varepsilon/2)(t - T)\ln^2 t \tag{3.114}$$

if $t > T$ for some large T. Since with this choice z will be a subsolution of equation (3.112) in $D = \{(x,t) : x \in \mathbb{R}, t > T\}$ and $z(x,T) = -\infty$, we conclude from the maximum principle that $v_{xx} \geq z$ in D hence the conclusion in the limit. □

The optimality of estimate (3.111) is checked by looking at the explicit solution $v_*(x,t;c)$, for which we have the estimate

$$(v_*)_{xx} \geq -(t\ln^2 t)^{-1} + O((t\ln^3 t)^{-1}) \quad \text{as } t \to \infty.$$

Observe that $(v_*)_{xx}(0,t;c) = -[t\ln^2(2t/c^2)]^{-1}$ for $t > c^2/2$.

Proof of Theorem 3.8. Under the additional assumption that u_0 has compact support, we now prove sharp estimates of the support of the solution

$$\operatorname{supp} u(x,t) = [s_-(t), s_+(t)],$$

which is a connected interval for large t, say for $t > t_1$.

We take the function $U(x,t) = u_*(x - d_+, t + T; a)$, where a is one-half of the mass of $u_0, T > a^2/2$ and $d_+ = s_+(t_1) + x_*(T; a)$, so that the support of $U(x,0)$, $(S_-(0), S_+(0))$, lies to the right of the support of u_0. Then, by the shifting comparison principle (see Section 2.5) we have a comparison of the interfaces of u and U, i.e., for $t > t_1$,

$$\begin{aligned} s_+(t) &\le S_+(t) = d_+ + x_*(t+T; a) \\ s_-(t) &\le S_-(t) = d_+ - x_*(t+T; a). \end{aligned} \tag{3.115}$$

A similar argument by shifting to the left gives $s_+(t) \ge -d_- + x_*(t+T; a)$ and $s_-(t) \ge -d_- - x_*(t+T; a)$, where $d_- = s_-(t_1) - x_*(T; a)$. In view of the formula for $x_*(t+T; a)$, we then have

$$s_+(t) = a \ln t + O(1), \quad s_-(t) = -a \ln t + O(1),$$

which completes the proof. □

Proof of Theorem 3.7. Consider the Cauchy problem (3.68), (3.69) for the quasilinear parabolic equation which is a perturbation of the autonomous one

$$\theta_\tau = \mathbf{A}(\theta). \tag{3.116}$$

By Lemma 3.13 the evolution orbit $\{\theta(\cdot,\tau), \tau > \tau_*\}$ is uniformly bounded, and hence, by a general regularity result for quasilinear parabolic equations, it is compact in $C_0(\mathbb{R})$. We now prove that the ω-limit set $\omega(\theta_0) = \big\{f \in C_0(\mathbb{R}) : f \ge 0$ and there exists $\{\tau_j\} \to \infty$ such that $\theta(\cdot,\tau_j) \to f(\cdot)$ as $j \to \infty$ uniformly in $\mathbb{R}\big\}$, is precisely

$$\omega(\theta_0) = \{F_a(\cdot)\}, \tag{3.117}$$

which indeed yields (3.60).

Choose an arbitrary $f \in \omega(\theta_0)$, so that there exists a sequence $\{\tau_j\} \to \infty$ such that

$$\theta(\cdot,\tau_j) \to f(\cdot) \quad \text{as } j \to \infty \text{ uniformly in } \mathbb{R}. \tag{3.118}$$

Applying Aleksandrov's reflection principle and passing to the limit $\tau \to \infty$, we have $f = f(|\eta|)$ and f does not increase in $|\eta|$.

We now prove that $f(a) = 0$. Suppose for a contradiction that $f(a) > 0$ and hence by continuity

$$\operatorname{meas}(\operatorname{supp} f) > 2a. \tag{3.119}$$

Using the conservation law (3.77) for the rescaled function θ with $\tau = \tau_j$, we obtain

$$\int_{-\infty}^{+\infty} \left[1 - \frac{\ln \theta(\eta, \tau_j)}{\ln \tau_j}\right]^{-1} d\eta \equiv E_0 = 2a \tag{3.120}$$

for $j = 1, 2, \ldots$. It follows from (3.118) that for a given small $\varepsilon > 0$, there exists $j_\varepsilon > 0$ such that

$$\theta(\cdot, \tau_j) \geq (f(\cdot) - \varepsilon)_+ \quad \text{in } \mathbb{R} \text{ for any } j > j_\varepsilon. \tag{3.121}$$

Therefore (3.119) and (3.121) imply that for any $\varepsilon > 0$ small enough,

$$\text{meas}\,(\text{supp}\,\theta(\cdot, \tau_j)) \geq \text{meas}\,(\text{supp}\,(f(\cdot) - \varepsilon)_+) > 2a \tag{3.122}$$

for $j > j_\varepsilon$. Combining (3.120)–(3.122) yields the estimate

$$\int_{-\infty}^{+\infty} \left[1 - \frac{\ln \theta(\eta, \tau_j)}{\ln \tau_j}\right]^{-1} d\eta \geq \int_{-\infty}^{+\infty} \left[1 - \frac{\ln((f(\eta) - \varepsilon)_+)}{\ln \tau_j}\right]^{-1} d\eta \tag{3.123}$$

$$\to \text{meas}\,(\text{supp}\,(f(\cdot) - \varepsilon)_+) > 2a \quad \text{as } j \to \infty,$$

contradicting the conservation law (3.120).

Thus, $f(a) = 0$ and meas (supp f) $\leq 2a$. Using Lemma 3.15, we deduce that

$$f(0) \geq a^2/2. \tag{3.124}$$

Rewriting estimate (3.111) for the function $\theta(\eta, \tau)$, integrating this inequality twice and passing to the limit $\tau = \tau_j \to \infty$, we obtain

$$f(\eta) \geq \tfrac{1}{2}(f(0) - \eta^2)_+. \tag{3.125}$$

By using (3.111) again, we may also conclude that $f_{\eta\eta} \geq -1$ a.e. Since $F_{\eta\eta}(\eta; a) \equiv -1$ in $[0, a]$, from (3.124) and (3.125) we have that the difference $z(\eta) = f(\eta) - F_a(\eta)$ satisfies $z \geq 0$, $z_{\eta\eta} \geq 0$ a.e. in $[0, a]$, and since $z(a) = 0$, one can see that $z_\eta(a) \leq 0$. Assume for contradiction that $z \not\equiv 0$ and hence $z_{\eta\eta} > 0$ in a set of nonzero measure in $[0, a]$. Then integrating the inequality $z_{\eta\eta} \geq 0$ over $(0, a)$ yields

$$z_\eta(0) < z_\eta(a) \leq 0,$$

contradicting the symmetry condition at the origin. This completes the proof. □

Comparison with the PME. It is interesting to consider our equation (3.55) as some kind of limit of the PME $u_t = (u^m)_{xx}$ as $m \to \infty$. Thus, the PME admits a family of explicit self-similar ZKB solutions denoted now by $\mathcal{U}_m(x, t)$ (see precise formulae in Section 2.2), which decay in time according to

$$\mathcal{U}_m(x, t) \leq a_m \|u_0\|_1^{2/(m+1)} t^{-1/(m+1)}, \quad a_m > 0, \tag{3.126}$$

while their support is confined by the interfaces $\pm s_m(t)$,

$$s_m(t) = c_m \|u_0\|_1^{(m-1)/(m+1)} t^{1/(m+1)}, \quad c_m > 0. \tag{3.127}$$

These estimates are true for every nonnegative solution while initial data are compactly supported. Put now $v = \mathcal{U}_m^m$ (in analogy to (3.58)) and let $m \to \infty$ to obtain a formal expression for the upper bound in the limit

$$v_\infty(x,t) \le a_\infty \|u_0\|_1^2 t^{-1}, \tag{3.128}$$

which agrees with Theorem 3.7. Agreement with Theorem 3.8 for the interfaces necessitates replacing $t^{1/(m+1)}$ in the limit $m = \infty$ by a slow growing factor $\ln t$ (and not by 1), then obtaining

$$s_\infty(t) = c_\infty \|u_0\|_1 \ln t. \tag{3.129}$$

Remarks and comments on the literature

General existence, uniqueness, comparison and regularity results for quasilinear filtration equations with arbitrary superslow diffusion coefficients can be found in the survey [202], see also the book [96], and one of the first papers on the weak theory for nonlinear heat equations is [257]. Friedman [121] proposed an interesting application of superslow diffusion equations to describe drying of photographic film. The model is further studied in [99].

§ 3.1. We follow the results of our paper [145]. A one-sided estimate on the solutions for equations of superslow diffusion has been derived in [115]. This estimate is not sharp enough to describe the asymptotic behaviour.

The idea of *approximate self-similar solutions*, i.e., those which do not satisfy the equation (and have extra symmetries) but describe its asymptotic properties, are quite fruitful in the asymptotic theory of equations of nonlinear heat conduction. See a survey in [166] and Section 6 in [286]. The second author has viewed the same issue as a form of *asymptotic simplification*, [305], which is an idea going back to the *reduced equation* in Prandtl's boundary layer theory [271]. See references in those works.

§ 3.2. For the Cauchy problem in superslow diffusion we follow [170]. It is remarkable that the family of asymptotic rescaled profiles is the same for both the Cauchy and the initial-boundary value problems. This is not true for the PME $u_t = (u^m)_{xx}$ for a fixed $0 < m < \infty$, see references in [202], [169] and in Chapter 4. Explicit solutions (3.79) were constructed in [221], see also [286], p. 79. About Aleksandrov's reflection principle (*method of moving planes*) [1], [2] in the theory of nonlinear parabolic and elliptic equations, see Chapt. 9 in [183], and Section 2.5.

In the proof of Lemma 3.15, a simple reflection analysis can be done as follows. Given $b \gg 1$, the difference of two solutions $w(x,t) = u(x,t) - u(2b-x,t)$, where $u(2b-x,t)$ is the solutions reflected in x relative to the point $x = b$, formally solves for $x > b, t > 0$ a linear parabolic equation, obtained by a standard linearization-like procedure, with the Dirichlet boundary condition $w(b,t) \equiv 0$. Since the initial data satisfy $w(x,0) = u_0(x) - u_0(2b-x) \equiv u_0(2b-x) \leq 0$ for any compactly supported data u_0 if $b \gg 1$, by the maximum principle $w(x,t) \leq 0$. Therefore $w(x,t)$ is nonincreasing in x at $x = b$, whence the monotonicity: $u(x,t)$ is nonincreasing in x at any $x = b \gg 1$.

In the proof of Lemma 3.15 we first apply the technique of intersection comparison, see Section 2.5. In Step 4 of the proof we use a technique from [155], [162], which shows that, given a "complete" family of particular exact solutions, via a small C^1-perturbation any inflection point can be transformed to at least three (transversal) points of intersection. This analysis does not need the general result on multiple zeros proved in [8] (see also [229]).

The semiconvexity estimate in Lemma 3.16 follows the ideas of [16]. The shifting comparison principle in the proof of Theorem 3.8 was introduced in [300]. Some similar results for more general quasilinear heat equations can be proved by intersection comparison, see [128] and [286], p. 245. This establishes the connection between the two approaches. Let us give some details. By construction, the solutions $u(x,t)$ and $U(x,t)$ at $t = 0$ have a unique intersection: $I(0) = 1$. Then $I(t) \leq 1$ for all $t > 0$. Therefore, the opposite inequality $s_+(t_1) > S_+(t_1)$ would mean that this intersection would disappear at the interface at some $t_2 \leq t_1$ so that $I(t_1) = 0$ and hence $u(x,t_1) \geq (\not\equiv)\, U(x,t_1)$. This is impossible since both solutions have the same mass (L^1-norm).

It is a typical argument of intersection comparison (to be used several times in the next chapters) of solutions having a *common* evolution property like same masses, momenta, blow-up or other singularity times, etc. Such a common property makes it possible to establish a *lower* bound on the intersection number, like $I(t) \geq 1$ in the present proof. Together with the *upper* bound ($I(t) \leq 1$) by the Sturm theorem, this establishes both bounds on the number of intersections and completes the geometric analysis via intersection comparison. In the present example we finally arrive at the equality $I(t) = 1$ for all $t > 0$, and this gives sharp estimates on the support (3.115) and other L^∞ estimates. See similar comments in Section 2.5.1.

The symmetrization argument, based on Aleksandrov's reflection principle, in the proof of Theorem 3.7 is given in Section 5 of [210]; see Section 2.5.3.

In the final remark, a rigorous limit $m \to \infty$ in the equation $u_t = (\Phi_m(u))_{xx}$ leads to the so called "mesa problem" studied by several authors, cf. [37], [67], [104], [122], [284]. There are other instances of mesa problems: a mesa problem for $m \to 0$ is described in [280].

4

Quasilinear Heat Equations with Absorption. The Critical Exponent

We present here the second example of application of the S-Theorem. We consider a quasilinear heat equation with two different operators, one representing diffusion, the other one absorption. We show that there exists a special critical relation of the exponents where these two operators generate a nontrivial nonlinear interaction which gives rise to an unusual asymptotic behaviour, more complex than the one corresponding to noncritical exponents.

Actually, we prove that the critical case produces an *extra logarithmic factor* that modifies the power-like decay and expansion rates that are valid for noncritical exponents bordering the critical case. As a result, the stable asymptotic pattern is not scaling invariant in the critical case we study here, though the equation with power nonlinearities admits a scale group in all cases.

The logarithmic factor occurs precisely when the a priori calculation of the decay and expansion produced by both mechanisms gives the same rates; it is therefore a clear sign that a phenomenon of *resonance* between the two physical effects is taking place. This is a quite general phenomenon, which can be studied by our stability technique. We still use the simpler version of the S-Theorem formulated for the stability condition (H3a), without using the reduced omega-limit sets.

We devote the final sections of the chapter to demonstrate the technique on two further examples of resonant behaviour for diffusion-absorption equations. We treat first the p-Laplacian (i.e., gradient-dependent) evolution equation with critical absorption exponent, and then the dipole-like solutions for diffusion-absorption in a half-line. These sections represent additional material and can be skipped in a first reading, or used as a source for student work.

4.1 Introduction: Diffusion-absorption with critical exponent

We investigate the asymptotic behaviour of the solution of the Cauchy problem for the PME with absorption

V. A. Galaktionov et al., *A Stability Technique for Evolution Partial Differential Equations*

$$u_t = \Delta u^{\sigma+1} - u^{\beta} \quad \text{in } Q = \mathbb{R}^N \times (0, \infty), \tag{4.1}$$

$$u(x, 0) = u_0(x) \quad \text{for } x \in \mathbb{R}^N. \tag{4.2}$$

We assume that $u_0 \not\equiv 0$ is integrable, nonnegative and compactly supported, the space dimension $N \geq 1$ and the exponents σ and β are positive. We will be concerned with the so-called *critical* value of the absorption exponent, $\beta_* = \sigma + 1 + 2/N$, or, better expressed, the critical relation between β and σ. In comparing with the notation used in the previous chapters, note that we use the notation $\sigma + 1$ for the diffusion exponent instead of m.

The asymptotic behaviour of the solution to problem (4.1), (4.2) under the above assumptions on u_0, σ and with $\beta \geq 1$ but $\beta \neq \beta_*$ is purely self-similar, has been established by scaling techniques, and does not need our methods. Namely, the following holds:

(i) For $\beta > \beta_*$ the effect of the absorption term $-u^{\beta}$ is negligible for large times, and the solution converges to one of the self-similar Zel'dovich–Kompaneetz–Barenblatt (ZKB) solutions of the PME $u_t = \Delta u^{\sigma+1}$. This phenomenon is called *asymptotic simplification* or *asymptotic degeneracy*, cf. [305] and [286], Chapter 6. As indicated in Section 2.2 the rates of decay and expansion for $t \gg 1$ are

$$u = O(t^{-k}), \qquad r(t) = O(t^{k/N}),$$

where $k = N/(N\sigma + 2)$, corresponding to the ZKB profiles.

(ii) For the range $\sigma + 1 < \beta < \beta_*$ both diffusion and absorption are involved in the asymptotics and the solution converges to the so-called self-similar *very singular solution* (VSS) of (4.1), which is uniquely defined. The exponents are now

$$u = O(t^{-\gamma}), \quad r(t) = O(t^{\mu}),$$

with $\gamma = 1/(\beta - 1)$ and $\mu = (\beta - \sigma - 1)/2(\beta - 1)$ are only possible similarity exponents for the whole equation, as the reader can easily check for himself. See further comments at the end.

We will show in this chapter that the asymptotic behaviour for the solutions of (4.1) in the critical case $\beta = \beta_*$ can be described in any space dimension $N \geq 1$ by means of a *unique contracted ZKB profile* corresponding to a total mass decreasing in time with a logarithmic rate. Before stating the main result, let us recall the notation: let $\mathcal{U}_M(x, t)$ be the ZKB solution to the PME (see Section 2.2)

$$u_t = \Delta u^{\sigma+1} \tag{4.3}$$

with mass $\int \mathcal{U}_M(x, t)\, dx = M > 0$, i.e.,

$$\mathcal{U}_M(x, t) = t^{-k} F(x t^{-k/N}; a), \tag{4.4}$$

where $k = N/(N\sigma + 2)$ and $F(\xi; a) = F_a(\xi)$ is given by the formula

$$F(\xi; a) = C_0 \left(a^2 - |\xi|^2\right)_+^{1/\sigma}, \quad C_0 = [k\sigma/2N(\sigma + 1)]^{1/\sigma}. \tag{4.5}$$

The parameter $a > 0$ is determined as a function of M to satisfy the mass condition $\int \mathcal{U}_M(x,t)\,dx = \int F(\xi;a)\,d\xi = M$, which means that

$$M = C_1 a^{N/k\sigma} \quad \text{with } C_1 = \pi^{N/2} C_0 B(N/2, 1+1/\sigma)/\Gamma(N/2)\,. \tag{4.6}$$

Here B and Γ denote, as usual, Euler's Beta and Gamma functions. With these notations, our main result can be formulated as follows.

Theorem 4.1 *Let $\sigma > 0$ and $\beta = \sigma + 1 + 2/N$. Then under the stated hypotheses on u_0, we have that as $t \to \infty$ the mass estimate*

$$\int u(x,t)\,dx = M_*(t)(1+o(1)) \quad \textit{with } M_*(t) = C_*(\ln t)^{-N/2}\,. \tag{4.7}$$

We also have precise pointwise estimate

$$u(x,t) = \mathcal{U}_{M_*(t)}(x,t) + o(\|\mathcal{U}_{M_*(t)}(\cdot,t)\|_\infty)$$

$$= (t \ln t)^{-k}[F_{a_*}(xt^{-k/N}(\ln t)^{k\sigma/2}) + o(1)]\,, \tag{4.8}$$

where the values of C_* and a_* are uniquely determined by $C_* = C_1\, a_*^{N/k\sigma}$ and

$$a_* = C_0^{-\sigma/2}\left[\frac{N\,B(N/2, 1+1/\sigma)}{2\,B(N/2, 1+\beta/\sigma)}\right]^{k\sigma/2}\,. \tag{4.9}$$

Remarks. We recall that for the purely diffusive PME (4.3), the total mass $M(t) = \int u(x,t)\,dx$ is conserved, while for equation (4.1) the mass is always a decreasing function of time, with a positive limit as $t \to \infty$ if $\beta > \beta_*$ and a zero limit if $\beta < \beta_*$. The decay rate is in the latter case power-like in t. Note that the critical decay and expansion rates are

$$u = O((t \ln t)^{-k}), \qquad r(t) = O(t^{k/N}(\ln t)^{-k\sigma/2})\,.$$

We also remark that the extra $\ln t$-scaling factors in Theorem 4.1 remain valid for the whole range of the diffusion exponent

$$\max\{-1, -2/N\} < \sigma < \infty,$$

where L^1-solutions of the PME preserve mass in time and the asymptotic behaviour corresponds to ZKB-like solutions (for $\sigma < 0$ they are not compactly supported, but they exhibit similar scaling structure, see Section 2.2 and comments).

OUTLINE OF THE PROOF OF THEOREM 4.1. It begins in Section 4.2 with an asymptotic estimate of the mass $M(t) = \int u(x,t)\,dx$ of a solution based on the assumption that u will behave for large times like a ZKB-function with the same mass. As the next important step, we construct in Section 4.3 some special weak sub- and supersolutions, which have the expected decay rate in u and growth rate in support and allow us to derive suitable lower and upper bounds for the solution.

We then introduce the rescaled variable

$$\theta(\xi,\tau) = ((T+t)\ln(T+t))^k u(\xi(T+t)^{k/N}(\ln(T+t))^{-k\sigma/2}, t), \qquad (4.10)$$

where $\tau = \ln(T+t)$ is the new time variable and T is a large constant. In terms of θ the asymptotic results of Theorem 4.1 just mean that $\int \theta(\xi,\tau)\,d\xi \to C_*$ and $\theta(\cdot,\tau)$ converges uniformly to F_{a_*} as $\tau \to \infty$. The equation for θ has the form

$$\theta_\tau = \mathbf{B}(\theta,\tau) \equiv \mathbf{A}(\theta) + \frac{1}{\tau}\mathbf{C}(\theta), \qquad (4.11)$$

where the autonomous part

$$\mathbf{A}(\theta) = \Delta\theta^{\sigma+1} + (k/N)(\nabla\theta\cdot\xi) + k\theta \qquad (4.12)$$

is the operator corresponding to the PME (after the natural rescaling, i.e., (4.10) without the log-terms). We have shown in Sections 2.3 and 2.4 that solutions in $L^1(\mathbb{R}^N)$ of the infinite-dimensional dynamical system $\theta_\tau = \mathbf{A}(\theta)$ converge as $\tau \to \infty$ to its equilibria (the ZKB profiles F_a). On the other hand, the operator $\mathbf{C}$ in the perturbation term

$$\mathbf{C}(\theta) = k\theta - (k\sigma/2)(\nabla\theta\cdot\xi) - \theta^\beta \qquad (4.13)$$

is a first-order operator. At this stage we apply the S-Theorem from Chapter 1 in order to show that the ω-limit set for solutions of equation (4.11) is a subset of the ω-limit set Ω_* for $\theta_\tau = \mathbf{A}(\theta)$.

The above convergence result allows us to return to the asymptotic estimate of Section 4.2 which can now be completely justified, thus selecting $\mathcal{U}_{M_*(t)}$ with $M_*(t) = C_*(\ln t)^{-N/2}$ as the correct asymptotics. Alternatively, we can perform the mass (or energy) analysis on the rescaled variable. Thus, in Section 4.5 we show that our ω-limit of $\theta(\tau)$ consists only of a single point, which is independent of the particular initial data taken by u and depends only on $\mathbf{C}$. In fact, the unique ω-limit is precisely the ZKB profile F_a for which $\int \mathbf{C}(F_a(\xi))\,d\xi = 0$. The selection property of $\mathbf{C}$ depends on the factor $1/\tau$ in (4.11) not being very "small". More precisely, we use the fact that $\int_1^\infty (d\tau/\tau)$ diverges and hence the perturbation is not *integrable*.

p-LAPLACIAN EQUATION. While part of the proof uses properties of second-order equations, like the maximum principle, the ω-limit analysis can be applied in very general circumstances. The whole method can be applied to another equation with critical exponent, namely to the *p-Laplacian equation with absorption* written in the form

$$u_t = \operatorname{div}(|Du|^\sigma Du) - u^\beta \qquad (4.14)$$

with $N \ge 1$, $\sigma > 0$ (actually, the results apply to the range $\sigma > -2/(N+1)$), same conditions on u_0; β takes on the critical value $\beta_* = \sigma + 1 + (\sigma+2)/N$. Here $Du = (u_{x_1}, ..., u_{x_N})$ is the spatial gradient of u and $|Du|$ stands for its length. The results are the same (see Section 4.6) with the only apparent difference in the value of the decay exponent k, which is now $N/[\sigma(N+1)+2]$. The similarity is recovered nevertheless when we observe that in both cases

$$k = 1/(\beta_* - 1), \tag{4.15}$$

which is the decay exponent of the purely absorptive equation $u_t = -u^\beta$ when $\beta = \beta_*$. Indeed, the critical value β_* is precisely determined in both equations as the one for which the diffusive and absorptive decay rates coincide and reinforce each other, thus giving rise to the extra $\ln t$ factors.

DIRAC MASSES. In Section 4.7 we apply our asymptotic estimates after a rescaling transformation to obtain insight into the question of nonexistence of solutions of (4.1) or (4.14) taking on a Dirac mass as initial trace, usually called source-type or fundamental solutions. We approximate the initial Dirac mass by a sequence of smooth functions with compact support ϕ_n obtained from a given ϕ by rescaling and show that, as a consequence of Theorem 4.1, the corresponding solutions converge uniformly to 0 as $n \to \infty$ in any region of the form $\mathbb{R}^N \times (t_0, \infty)$ with $t_0 > 0$, thus giving rise to an *initial layer* across which the solution "disappears". It will be apparent from the proof that it is precisely the existence of the extra $\ln t$ factors in the expressions for large t that implies this phenomenon for small t.

MORE GENERAL DATA. We end our study of the asymptotic behaviour for solutions of (4.1) (or (4.14)) by considering what happens when the restriction of compact support is eliminated. Rather to our surprise, we discovered that there are solutions with integrable initial data for which *no* $\ln t$ *factors appear* in the decay rates and the best estimates that may be obtained are no better than those of the diffusive equation.

Theorem 4.2 *For any solution $u(x,t)$ of* (4.1) *or* (4.14) *with critical exponent β_* and such that $u(x,0) \in L^1(\mathbb{R}^N)$, we have as $t \to \infty$,*

$$M(t) = \int u(x,t)\,dx \to 0 \quad \textit{and} \tag{4.16}$$

$$u(x,t)t^k \to 0 \quad (k = 1/(\beta_* - 1)) \tag{4.17}$$

uniformly in $x \in \mathbb{R}^N$. These rates cannot be improved under the stated assumptions.

The proofs rely strongly on the use of rescaling transformations. The argument is also valid for the semilinear case $\sigma = 0$, i.e., equation $u_t = \Delta u - u^\beta$ with $\beta = (N+2)/N$ and $k = N/2$.

EXTENSIONS. The results obtained for equation (4.1) can be easily generalized to other similar equations, for instance to

$$u_t = \Delta u^m - f(u) \tag{4.18}$$

under an assumption of *critical behaviour* on the nondecreasing function $f : [0,\infty) \to [0,\infty)$ of the form $f(u)/u^{\beta_*} \to c \in (0,\infty)$ as $u \to 0$. Further details are given in Section 4.9. The same applies to equation (4.14).

HALF-SPACE, PATTERNS AND RATES. In the last Section 4.10 we describe a problem where a similar resonance situation occurs and similar techniques and results apply. We study the asymptotic patterns for the one-dimensional PME with absorption posed in a half-space, and derive the corresponding $\ln t$-perturbed behaviour in the critical case.

4.2 First mass analysis

In what follows we put $\beta = \beta_*$. We consider the solution of the Cauchy problem for the quasilinear heat equation with absorption (4.1) with initial data (4.2), under the assumptions stated above. The existence and uniqueness of a weak solution $u \in C([0,\infty) : L^1(\mathbb{R}^N))$ which is nonnegative, continuous for $t > 0$ and has compact support in the space variable for every $t > 0$, as well as comparison theorems for super- and subsolutions of the problem, are well known; see comments at the end of the chapter.

Let us make a formal estimate of the evolution of the total mass $M(t)$ on the assumption that for every fixed and large time, the solution $u(t) = u(x,t)$ can be uniformly approximated by the ZKB-function with the same mass $M(t) = \int u(x,t)\,dx$:

$$u(\cdot,t) = \mathcal{U}_{M(t)}(\cdot,t) + o(\|\mathcal{U}_{M(t)}(t)\|_\infty)\,, \tag{4.19}$$

while the support of $u(t)$ is contained in a fixed multiple of the support of $\mathcal{U}_{M(t)}$. We begin by integrating equation (4.1) over $\mathbb{R}^N$ to obtain

$$\frac{d}{dt}M(t) = \int u_t\,dx = \int (\Delta u^{\sigma+1})dx - \int u^\beta\,dx\,. \tag{4.20}$$

The first term on the right-hand side vanishes for solutions with compact support, while in order to estimate the integral $\int u^\beta\,dx$ for large $t > 0$, we compute

$$\begin{aligned}\int \mathcal{U}^\beta_{M(t)}\,dx &= t^{-\beta k}C_0^\beta \int \left(a^2 - |x|^2 t^{-2k/N}\right)_+^{\beta/\sigma} dx \\ &= t^{-1}C_0^\beta a^{N+2\beta/\sigma}\int (1-|\xi|^2)_+^{\beta/\sigma}\,d\xi = C_2 a^{N+2\beta/\sigma}t^{-1}\,,\end{aligned} \tag{4.21}$$

where $a = a(t)$ is related to $M(t)$ by (4.6) and

$$C_2 = \pi^{N/2}C_0^\beta B(N/2, 1+\beta/\sigma)/\Gamma(N/2)\,. \tag{4.22}$$

Combining (4.6), (4.19), (4.20) and (4.21), we get for $\beta = \beta_*$,

$$\frac{d}{dt}M(t) = -C_3 M^{1+2/N}(t)t^{-1}(1+o(1)) \tag{4.23}$$

with $C_3 = C_2 C_1^{-(1+2/N)}$, integration of which gives for $t \gg 1$

$$M(t) = C_*(\ln t + K)^{-N/2}(1+o(1)) = C_*(\ln t)^{-N/2}(1+o(1))\,, \tag{4.24}$$

where $C_* = (2C_3/N)^{-N/2} = C_1\,(NC_1/2C_2)^{N/2}$ and K is the integration constant. In this way we formally arrive at estimate (4.7) of Theorem 4.1, and (4.8) would follow from it. The function

$$w(x,t) = (t\ln t)^{-k} F(xt^{-k/N}(\ln t)^{k\sigma/2}; a_*)\,, \tag{4.25}$$

which can replace the solution $u(x,t)$ to first-order of approximation, is not a solution of (4.1) but its *approximate* self-similar solution. It is an explicit solution of a different nonautonomous quasilinear parabolic equation with a linear absorption term, namely

$$w_t = \left(1 - \frac{N\sigma}{2\ln t}\right) \Delta w^{\sigma+1} - \frac{Nw}{2t\ln t}\,. \tag{4.26}$$

4.3 Sharp lower and upper estimates

The first step in the rigorous proof of Theorem 4.1 consists in obtaining upper and lower estimates with exact growth or decay rates. We briefly describe the main steps in the proof. A similar but more delicate and detailed analysis is presented in Section 4.10.

Lemma 4.3 *For any $T > 1$ and $0 < a < A = C_0^{-\sigma/2} k^{k\sigma/2}$, the function*

$$\underline{u}(x,t;T,a) = ((T+t)\ln(T+t))^{-k} F(\xi; a) \tag{4.27}$$

is a weak subsolution of equation (4.1) *in Q.*

Proof. By weak subsolution we understand a continuous, nonnegative function u which satisfies the integral inequality

$$\iint u\phi_t + \iint u^{\sigma+1}\Delta\phi - \iint u^\beta \phi \ge 0 \tag{4.28}$$

for every test function $\phi \in C_0^\infty(Q)$, $\phi \ge 0$. One can see that our function $\underline{u} \ge 0$, $\underline{u}^\sigma$ is Lipschitz continuous and that for $a \in (0, A)$ the inequality

$$\underline{u}_t \le \Delta \underline{u}^{\sigma+1} - \underline{u}^{\sigma+1+2/N} \tag{4.29}$$

holds at every point of Q where $\underline{u} > 0$. This is enough for $\underline{u}$ to be a subsolution, as the reader may easily check. □

Lemma 4.4 *For sufficiently large $T > e^{1+N\sigma}$ and constants $a > 0$ and $b > N\sigma/[2(1-(1+N\sigma)/\ln T)] > 0$, the function*

$$\begin{gathered}\overline{u}(x,t;T,a) = ((T+t)\ln(T+t))^{-k} F(\phi(t)\xi; a)\,,\\ \textit{with}\ \ \phi(t) = [1 + b/\ln(T+t)]^{-1/2},\end{gathered} \tag{4.30}$$

is a weak supersolution of (4.1) *in Q.*

Proof. The definition of weak supersolution is similar to (4.28) with $\geq$ replaced by $\leq$. As in Lemma 4.3, the proof consists in checking that (4.29) holds with reverse inequality at points where $\overline{u} > 0$, since $\overline{u}^\sigma$ is Lipschitz continuous. Further details can be found in the proof of Lemma 4.20. □

Lemma 4.5 *There exist constants a_- and a_+, $0 < a_- < a_+$, such that for $x \in \mathbb{R}^N$ and $t > T \gg 1$,*

$$\begin{aligned} ((T+t)\ln(T+t))^{-k} F(\xi; a_-) &\leq u(x,t) \\ &\leq ((T+t)\ln(T+t))^{-k} F(\xi; a_+). \end{aligned} \tag{4.31}$$

Proof. There exists $T_1 > 0$ such that $u(0, T_1) > 0$ and $u(0, T_1)$ is a continuous function with compact support. Consider $u(x,t)$ as the solution of the Cauchy problem for (4.1) in $\mathbb{R}^N \times (T_1, \infty)$ with compactly supported initial function $u(x, T_1)$. By Lemma 4.4 and standard comparison of weak sub- and supersolutions, we can find a value of $a = a_1$, such that for fixed and large enough $T_2 > 2T_1 + 1$,

$$u(x,t) \leq \overline{u}(x,t; T_2 - T_1, a_1) \quad \text{in } \mathbb{R}^N \times (T_1, \infty). \tag{4.32}$$

Let $T_2 - T_1 = T$. Then

$$u(x,t) \leq ((T+t)\ln(T+t))^{-k} F(\xi/[1 + b/\ln(T+t)]^{1/2}; a_1)$$

for $t > T_1$. Using the fact that $F(\xi; a)$ is a nonincreasing function of $|\xi|$, we obtain

$$\begin{aligned} F(\xi/[1 + b/\ln(T+t)]^{1/2}; a_1) &\leq F(\xi/[1 + b/\ln T]^{1/2}; a_1) \\ \equiv [1 + b/\ln T]^{-1/\sigma} F(\xi; a_1[1 + b/\ln T]^{1/2}) &\leq F(\xi; a_1[1 + b/\ln T]^{1/2}). \end{aligned}$$

This implies the upper estimate in (4.31) with $a_+ = a_1[1 + b/\ln T]^{1/2}$.

In order to prove the lower estimate, we take for a fixed T as above a constant $a_- > 0$ so small that

$$u(x,T) \geq [2T\ln(2T)]^{-k} F(x(2T)^{-k/N}[\ln(2T)]^{k\sigma/2}; a_-) \equiv \underline{u}(x, T; T, a_-). \tag{4.33}$$

It then follows from Lemma 4.3 that $u(x,t) \geq \underline{u}(x,t)$ for every $t \geq T$. □

If we now perform the change of variables (4.10), the function θ will be a weak solution of the equation (4.11) taking at $\tau_* = \ln T$ the initial condition

$$\theta(\xi, \tau_*) = (T \ln T)^k u_0(\xi T^{k/N} (\ln T)^{-k\sigma/2}) \tag{4.34}$$

for $\xi \in \mathbb{R}^N$. By Lemma 4.5, θ is bounded from above and below in $\mathbb{R}^N \times (\tau_0, \infty)$, $\tau_0 = \ln(2T)$:

$$F(\xi; a_-) \leq \theta(\xi, \tau) \leq F(\xi; a_+). \tag{4.35}$$

As a consequence of these estimates, we can also control the growth of the support of $u(\cdot, t)$ as $t \to \infty$.

Corollary 4.6 *For every solution as above, we have the following estimate for the support: there exists T such that for $t > T$,*

$$\{|x| \le a_-(T+t)^{k/N}(\ln(T+t))^{-k\sigma/2}\} \subseteq \text{supp}\,(u(\cdot,t))$$

$$\subseteq \{|x| \le a_+(T+t)^{k/N}(\ln(T+t))^{-k\sigma/2}\}\,. \tag{4.36}$$

4.4 ω-limits for the perturbed equation

We apply the S-Theorem to the equation satisfied by θ, (4.11)–(4.13). To adapt the notation of Sections 1.1–1.3 to our equation, we go back to the terminology and values of Section 4.3 and make the correspondences

$$t = \tau, \quad x = \xi \in B = B_{a_+}(0) \subset \mathbb{R}^N, \quad u = \theta, \tag{4.37}$$

with a_+ defined in Lemma 4.5. We take as $\mathbf{A}$ the operator defined in (4.12) with $\mathbf{C}$ given by (4.13). As a functional space we take

$$X = \{f \in L^1(B) : F(\xi; a_-) \le f(\xi) \le F(\xi; a_+) \quad \text{a.e. in } B\} \tag{4.38}$$

with $F(\xi; a)$ defined in (4.5).

Let us check that $\mathbf{A}$ satisfies the condition (H3a) in Section 1.5. The PME (4.3) is well known to generate a semigroup of contractions in $L^1(\mathbb{R}^N)$. In fact, for every two solutions θ_1 and θ_2 of the rescaled PME

$$\theta_\tau = \mathbf{A}(\theta) \tag{4.39}$$

with initial data $\theta_1(\cdot, 0), \theta_2(\cdot, 0)$ in $L^1(\mathbb{R}^N)$ and every $\tau > 0$, we have

$$\int [\theta_1(\xi,\tau) - \theta_2(\xi,\tau)]_+ \, d\xi \le \int [\theta_1(\xi,0) - \theta_2(\xi,0)]_+ \, d\xi\,. \tag{4.40}$$

This is the T-contraction property, cf. (2.20). In particular, the L^1-norm of any non-negative solution is an invariant of the evolution. The standard comparison result that follows from (4.40) ensures that all solutions with initial data in X stay in X for all positive times. On the other hand, the ZKB profiles represent the equilibria of $\theta_\tau = \mathbf{A}(\theta)$ in the above class X.

For the PME, the ω-limit set of any solution with initial datum $\theta_0 \in X$ consists precisely of the ZKB profile $F(\xi; a)$ with the same L^1-norm as θ_0 (this uniquely determines the constant a). Therefore, the reduced ω-limit set Ω_* of the autonomous equation (4.39) consists in our application of

$$\Omega_* = \{F(\xi; a) : a_- \le a \le a_+\}\,, \tag{4.41}$$

which is clearly a compact subset of X. In fact, it consists exclusively of fixed points and the L^1-contraction property (4.40) implies that Ω_* is uniformly Lyapunov stable. Moreover, every point of Ω_* is Lyapunov stable (with $\varepsilon = \delta$ in the definition (H3a)).

As for condition (H1) in Section 1.3, the solutions to $\theta_\tau = \mathbf{B}(\theta, \tau)$ stay in X thanks to a similar comparison argument, cf. Lemma 4.5. It follows from the boundedness of the orbit in $L^\infty(\mathbb{R}^N)$ together with general interior regularity results for parabolic equations [96] that the orbits are relatively compact in the space of continuous functions $C(B)$.

Finally, one easily checks in the definition of weak solution that, due to the boundedness of θ, the integral terms coming from $(1/\tau)\mathbf{C}(\theta)$ converge to 0 as $\tau \to \infty$, which proves (H2).

Consequently, the S-Theorem can be applied. It should be noted that the convergence takes place not only in $L^1(B)$ but also in the uniform norm as a consequence of the interior regularity theory. This is an important fact that we shall use in the next section.

4.5 Extended mass analysis: Uniqueness of stable asymptotics

The end of the proof of Theorem 4.1 is now straightforward. By the S-Theorem, we may approximate a solution $\theta(\cdot, \tau)$ of equation (4.11) by a ZKB profile F with a small uniform error if τ is large enough:

$$\theta(\cdot, \tau) = F(\cdot; a(\tau)) + \mu(\cdot, \tau) \tag{4.42}$$

with $\mu(\cdot, \tau) = o(1)$ uniformly in $\mathbb{R}^N$ as $\tau \to \infty$. We also know by Corollary 4.6 that the support of the solution can be estimated up to a multiplicative factor. It is then easy to prove that we may approximate $\theta(\cdot, \tau)$ by the profile F with same integral as $\theta(\cdot, \tau)$, i.e., we may choose $a(\tau)$ such that $\int F(\xi; a(\tau))\, d\xi = \int \theta(\xi, \tau)\, d\xi$ and the order of error will not be changed. Undoing the change of variables (4.10), we obtain

Proposition 4.7 *For large t, we have the uniform estimate*

$$u(\cdot, t) = \mathcal{U}_{M(t)}(\cdot, t) + o((t \ln t)^{-k}), \tag{4.43}$$

where $M(t)$ is the integral of u at time t.

Since the assumption on the support also holds, the asymptotic estimate for $M(t)$ obtained in Section 4.2 is justified. Together with the S-Theorem and the convergence results of Section 4.4, it proves Theorem 4.1.

We will give below a different proof of the existence of a unique asymptotic profile based on the study of the rescaled equation (4.11), since it gives a new light on this phenomenon and could be applicable to quite general equations of the form (4.11) under certain structural assumptions. We now study the evolution of the *rescaled mass* (or energy) of θ:

$$E(\tau) = \|\theta(\cdot, \tau)\|_1, \quad \tau > \tau_0. \tag{4.44}$$

By the regularity properties of the weak solution, we get a $u \in C^1(\mathbb{R}_+ : L^1(B))$, and hence $\theta \in C^1((\tau_0, \infty) : L^1(B))$. Moreover, it is easy to check that for every $\tau > \tau_0$,

$$\int \mathbf{A}(\theta(\xi,\tau))\,d\xi = 0 \qquad \text{and} \tag{4.45}$$

$$H(\theta(\tau)) \equiv \int \mathbf{C}(\theta(\xi,\tau))\,d\xi = (N/2)\|\theta(\cdot,\tau)\|_1 - \|\theta(\cdot,\tau)\|_\beta^\beta \tag{4.46}$$

with $\beta = \sigma + 1 + 2/N$. From (4.11)–(4.13) we have the following rescaled *mass equation* for $E(\tau)$:

$$\frac{dE}{d\tau} = \frac{1}{\tau}\,H(\theta(\tau)) \tag{4.47}$$

for $\tau > \tau_0$. Note that, by the estimates of Section 4.3, the trajectories for our equation have uniformly bounded energy and $H(\theta)$ is also bounded. The following asymptotic result completes the proof of Theorem 4.1.

Proposition 4.8 *Under the assumed hypotheses on* u_0

$$\omega(\theta_0) = \{F_{a_*}\}, \tag{4.48}$$

where a_* *is defined in* (4.9).

Proof. It is again based on the idea that for large τ the solution $\theta(\cdot,\tau)$ lies very close to the set $\Omega_* = \{F_a(\cdot) : a_- \le a \le a_+\}$. Hence we may replace θ by F_a, with $a = a(\tau)$, in formula (4.46) above and study the evolution of the corresponding ODE for $a(\tau)$. To this effect we shall use the formulas

$$\|F_a\|_1 = C_1\,a^{N/k\sigma}, \tag{4.49}$$

with C_1 as in (4.6), and

$$H(F_a) = C_2\,a^{N/\sigma k}\left(a_*^{2/\sigma k} - a^{2/\sigma k}\right) \tag{4.50}$$

with C_2 given by (4.22). Substituting $\|F_a\|_1$ for E and $H(F_a)$ for $H(\theta)$ in (4.47), we get the approximate equation

$$\frac{da}{d\tau} = \frac{C_4 a}{\tau}\left(a_*^{2/\sigma k} - a^{2/\sigma k}\right), \tag{4.51}$$

which has as only possible bounded stable asymptotics $a = a_*$. This approximate asymptotic calculation is justified in the following way.

Lemma 4.9 *Assume that there exists a limit*

$$E(\tau) \to E_0 \quad \textit{as } \tau \to \infty. \tag{4.52}$$

Then E_0 *equals* $\|F_{a_*}\|_1 = E_*$, *and* $\theta(\xi,\tau) \to F_{a_*}(\xi)$ *uniformly in* ξ.

Proof. (i) First, by (4.35) we have

$$0 < E_- \le E(\tau) \le E_+ \quad (E_\pm = \|F(\cdot; a_\pm)\|_1) \tag{4.53}$$

for all $\tau > \tau_0$, so that $E_0 \in [E_-, E_+]$. Under the assumption (4.52) it follows from the application of the S-Theorem to our equation that

$$\omega(\theta_0) \subseteq \{F_a : a_- \le a \le a_+, \ \|F_a\|_1 = E_0\}. \tag{4.54}$$

Since $\|F_a\|_1$ is a strictly increasing function of $a > 0$, cf. (4.49), the equation $\|F_a\|_1 = E_0$ has a unique solution $a = a_0$, $a_- \le a_0 \le a_+$, and $\omega(\theta_0)$ consists of a unique function F_{a_0}. Hence,

$$\theta(\cdot, \tau) \to F(\cdot; a_0) \quad \text{uniformly in } B \text{ as } \tau \to \infty. \tag{4.55}$$

(ii) Let us show that $a_0 = a_*$. Using (4.55), we have

$$H(\theta(\tau)) \to H(F(\cdot; a_0)) \tag{4.56}$$

as $\tau \to \infty$. By (4.50) the function $H(F_{a_0})$ is positive for $0 < a_0 < a_*$ and negative for $a_0 > a_*$. If $a_0 \ne a_*$, suppose, for instance, that $a_0 < a_*$. Then for τ large enough we can conclude from (4.47) and (4.56) that for any sufficiently large τ, the following inequality holds:

$$\frac{dE}{d\tau} \ge \frac{1}{2\tau} H(F_{a_0}) > 0. \tag{4.57}$$

Thus, $E(\tau) \to \infty$ as $\tau \to \infty$ which contradicts (4.53), i.e., the boundedness from above of the mass trajectory. We observe that the nonintegrability of $1/\tau$ at ∞ is crucial at this stage.

In the same way the opposite assumption $a_0 > a_*$ leads to the conclusion $E(\tau) \to -\infty$ as $\tau \to \infty$, in contradiction also with (4.53). □

In order to finish the proof of Proposition 4.8, we consider the possibility that (4.48) does not hold. Then from Lemma 4.9, we conclude that $E(\tau)$ cannot have a limit as $\tau \to \infty$. Since $E(\tau)$ is bounded, it is necessarily an oscillatory function near $\tau = \infty$, and hence by the compactness of the trajectory $\{\theta(\cdot, \tau), \ \tau > \tau_0\}$ (see Section 4.4) and the S-Theorem there exist two sequences $\{\tau_j\} \to \infty$ and $\{\tilde{\tau}_j\} \to \infty$ such that $\theta(\cdot, \tau_j) \to F(\cdot; a)$ and $\theta(\cdot, \tilde{\tau}_j) \to F(\cdot; a')$ uniformly in B as $j \to \infty$, where $a \ne a'$ and $a, a' \in [a_-, a_+]$. From (4.49) $\|F_a\|_1$ and $\|F_{a'}\|_1$ must be different. Then either $a \ne a_*$ or $a' \ne a_*$.

Consider, for instance, the case $a_* < a'$ and $a < a'$. Fix an arbitrary $a_1 \in (a, a')$, $a_1 > a_*$. Then, by the continuity of $E(\tau)$ and its oscillatory property near $\tau = \infty$ there exists a sequence $\tau_j' \to \infty$ as $j \to \infty$ such that not only $\|\theta(\tau_j')\|_1 = \|F_{a_1}\|_1$ but also the mass is nondecreasing at those points:

$$\frac{dE}{d\tau}(\tau_j') \ge 0 \quad \text{for all } j. \tag{4.58}$$

Then the S-Theorem implies that

$$\theta(\tau_j') \to F_{a_1} \quad \text{uniformly in } B \text{ as } j \to \infty, \tag{4.59}$$

and from the energy equation (4.47) we obtain

$$\tau_j' \frac{dE}{d\tau}(\tau_j') = H(\theta(\tau_j')) \to H(F_{a_1}) \quad \text{as } j \to \infty. \tag{4.60}$$

Now, since $a_1 > a_*$, it follows from (4.50) that $H(F_{a_1}) < 0$. Hence, from (4.60) we obtain the inequality

$$\tau_j' \frac{dE}{d\tau}(\tau_j') < 0 \quad \text{for all sufficiently large } j,$$

which contradicts (4.58). This assertion completes the proof of Proposition 4.8 and hence of Theorem 4.1. □

On a general concept of the extended mass analysis. The above analysis is based on some general arguments that we sum up as follows: we have a dynamical system (∗) $u_t = \mathbf{A}(u)$ and a perturbation (∗∗) $u_t = \mathbf{B}(u, t)$ with the usual properties (H1)–(H3a) from Section 4.4. We assume that there is a functional E, which remains bounded on orbits of (∗∗), is differentiable on the orbits and

$$\frac{dE(u(t))}{dt} = f(t)H(u(t)), \tag{4.61}$$

where H is a continuous function defined in a subspace X' of X which contains the orbits and Ω_* (in our case $X = L^1(B) \cap C(B)$). We also assume that the orbits are relatively compact in X', that the ω-limit set Ω_* of (∗) is a *linear set* (a one-dimensional manifold), i.e., it can be continuously parameterized in a one-to-one way with a parameter $a \in [a_-, a_+]$. Let H have only one zero on Ω_* and

$$\int^{\infty} f(t)\, dt = \infty. \tag{4.62}$$

Then the ω-limit of every solution is included in the zero of H in Ω_*. It should be interesting to obtain a general result without the very strict assumption on the topology of Ω_*.

4.6 Equation with gradient-dependent diffusion and absorption

Main result. As a second example of application of the techniques discussed so far, we study in this section the large-time behaviour of the solutions $u \geq 0$ of the p-Laplacian equation with absorption

$$u_t = \operatorname{div}(|Du|^{\sigma} Du) - u^{\beta} \quad \text{in } Q, \tag{4.63}$$

with $\sigma > 0$ and critical value for β, $\beta_* = \sigma + 1 + (\sigma + 2)/N$. The initial data $u(x, 0) = u_0(x)$ satisfy

$$u_0 \geq 0, \quad u_0 \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N), \quad u_0 \text{ has compact support.} \tag{4.64}$$

The existence, uniqueness and comparison results for nonnegative weak solutions to the Cauchy problem are well established; see comments at the end of the chapter.

The aim of this section is to describe the asymptotic behaviour of such solutions which again will turn out to be independent of the initial data. Indeed, we will show that $u(x, t)$ converges as $t \to \infty$ to the unique *approximate* self-similar solution

$$w(x, t) = (t \ln t)^{-k} F(\xi; a_*), \quad \xi = x\, t^{-k/N} (\ln t)^{k\sigma/(\sigma+2)}, \tag{4.65}$$

where the exponent k is now given by $k = N/[\sigma(N + 1) + 2] = 1/(\beta_* - 1)$, and

$$F_a(\xi) = F(\xi; a) = C_0 \left[a^{(\sigma+2)/(\sigma+1)} - |\xi|^{(\sigma+2)/(\sigma+1)} \right]_+^{(\sigma+1)/\sigma}, \tag{4.66}$$

$$C_0 = \left[\sigma(\sigma + 2)^{-1} (k/N)^{1/(\sigma+1)} \right]^{(\sigma+1)/\sigma}, \tag{4.67}$$

and $a_* = a_*(\sigma, N) > 0$ is the constant which is defined below. Function (4.4), (4.66) is a natural analogy of the ZKB solution of the PME for the p-Laplacian equation, see Section 2.2. In the next theorem the convergence of $u(x, t)$ to $w(x, t)$ as $t \to \infty$ is stated in terms of the rescaled function

$$\theta(\xi, t) = ((T + t) \ln(T + t))^k \; u(\xi (T + t)^{k/N} (\ln(T + t))^{-k\sigma/(\sigma+2)}, t) \tag{4.68}$$

for some large $T > 0$. The result, similar to Theorem 4.1, will be formulated below in the following equivalent way.

Theorem 4.10 *For any fixed $T > 1$, the following estimate holds:*

$$\theta(\xi, t) \to F(\xi; a_*) \quad \text{as } t \to \infty \tag{4.69}$$

uniformly in $\mathbb{R}^N$, where

$$a_* = \left[\frac{N C_0^{-1/k}\, B\left(\frac{(\sigma+1)N}{\sigma+2}, \frac{\sigma+1}{\sigma} + 1\right)}{(\sigma + 2)\, B\left(\frac{(\sigma+1)N}{\sigma+2}, \frac{(\sigma+1)\beta}{\sigma} + 1\right)} \right]^{k\sigma/(\sigma+2)} \tag{4.70}$$

This implies a decay estimate for the total mass as $t \to \infty$ of the form

$$\int u(x, t)\, dx = C_* (\ln t)^{-N/(\sigma+2)} (1 + o(1)), \tag{4.71}$$

where $C_* = C_1 a_*^{N/k\sigma}$ and C_1 given below in (4.83). This is completely similar to estimate (4.7) for equation (4.1).

As in the previous case, the proof consists of three steps. We begin with a preliminary result on the asymptotic behaviour.

Sharp lower and upper estimates.

Lemma 4.11 *There exist positive constants $a_- < a_+$ such that for sufficiently large $T > 1$, the following estimates hold in $\mathbb{R}^N \times (T, \infty)$:*

$$((T+t)\ln(T+t))^{-k} F(\xi; a_-) \le u(x,t) \le ((T+t)\ln(T+t))^{-k} F(\xi; a_+). \tag{4.72}$$

These estimates show that the approximate self-similar solution (4.65) describes the actual space-time structure of $u(x,t)$ for large t. As above, the proof is based on straightforward computations. See a similar more detailed construction for the dipole-like behaviour in Section 4.10.

Structure of the ω-limit set for the rescaled equation. The function $\theta(\xi, \tau)$, $\tau = \ln(T+t)$, defined in (4.68), satisfies the degenerate parabolic equation

$$\theta_\tau = \mathbf{B}(\theta, \tau) \equiv \mathbf{A}(\theta) + \frac{1}{\tau}\mathbf{C}(\theta) \tag{4.73}$$

in $B \times (\tau_*, \infty)$, where $B = \{|\xi| < a_+\}$ and $\tau_* = \ln T$. Here

$$\mathbf{A}(\theta) = \mathrm{div}(|D\theta|^\sigma D\theta) + (k/N)(D\theta \cdot \xi) + k\theta \tag{4.74}$$

is the autonomous part of the operator in (4.73) corresponding to the natural rescaling of the p-Laplacian equation $u_t = \mathrm{div}(|Du|^\sigma Du)$, while

$$\mathbf{C}(\theta) = k\theta - [k\sigma/(\sigma+2)](D\theta \cdot \xi) - \theta^\beta. \tag{4.75}$$

For any fixed $a > 0$, the function $F(\xi; a)$ from (4.66) is a radial weak solution with compact support of the stationary equation $\mathbf{A}(F) = 0$ in $\mathbb{R}^N$.

It follows from Lemma 4.11 that

$$F(\xi; a_-) \le \theta(\xi, \tau) \le F(\xi; a_+) \quad \text{in } \mathbb{R}^N \times (\tau_0, \infty) \tag{4.76}$$

with $\tau_0 = \ln(2T)$. Hence $\theta(\xi, \tau)$ satisfies the boundary condition

$$\theta(\xi, \tau) = 0 \quad \text{on } \partial B \text{ for any } \tau \ge \tau_0. \tag{4.77}$$

It also satisfies the initial condition

$$\theta(\xi, 0) = \theta_0(\xi) \quad \text{in } B, \tag{4.78}$$

where θ_0 has compact support in B.

Using (4.76), we conclude that the trajectory $\{\theta(\cdot, \tau), \tau > \tau_0\}$ is compact in $C(B)$, and we can define the ω-limit set $\omega(\theta_0)$ for the solution $\theta(\xi, \tau)$ in the space X given again by formula (4.38). We now apply the S-Theorem with the definitions (4.37), (4.38) and our present definitions of θ, ξ, τ, $\mathbf{A}$, $\mathbf{C}$, k, a_+, a_- and T. The necessary properties of the present operators $\mathbf{A}$ and $\mathbf{C}$ are similar to those described in Section 4.4 for equation (4.11)–(4.13). Thus, it is well known that the p-Laplacian operator generates a semigroup of (ordered) contractions in $L^1(\mathbb{R}^N)$, that the integral $\int u(x,t)\,dx$ of its solutions is invariant in time and that the rescaled solutions converge uniformly to ZKB-like profiles defined now by formula (4.66). The relative compactness of the orbits (even in the space $C(B)$) follows from standard interior regularity results for the quasilinear parabolic equations. Finally, condition (H2) is checked as in Section 4.4.

Extended mass analysis. As in Section 4.5, we obtain for $E(\tau) = \|\theta(\cdot,\tau)\|_1 \in C^1(\tau_0,\infty)$ the rescaled mass equation

$$\frac{dE}{d\tau} = \frac{1}{\tau} H(\theta(\tau)), \qquad \tau > \tau_0, \tag{4.79}$$

where

$$H(\theta) = \int \mathbf{C}(\theta(\xi,\tau))\,d\xi \equiv \frac{N}{\sigma+2}\|\theta(\cdot,\tau)\|_1 - \|\theta(\cdot,\tau)\|_\beta^\beta. \tag{4.80}$$

Again as in Proposition 4.8, we are able to show that for every solution

$$\omega(\theta_0) = \{F_{a_*}\}, \tag{4.81}$$

thanks to the following estimate of $H(\theta)$ on the set $\Omega_* = \{F_a : a_- \le a \le a_+\}$:

$$H(F_a) = C_2\, a^{N/k\sigma}\left(a_*^{(\sigma+2)/\sigma k} - a^{(\sigma+2)/\sigma k}\right), \tag{4.82}$$

$$C_2 = \frac{2\pi^{N/2}(\sigma+1)}{(\sigma+2)\Gamma(N/2)} C_0^{1+1/k} B\left(\frac{(\sigma+1)N}{\sigma+2}, \frac{(\sigma+1)\beta}{\sigma}+1\right)$$

and the fact that

$$\|F_a\|_1 = C_1\, a^{N/k\sigma}, \tag{4.83}$$

where

$$C_1 = 2\pi^{N/2} C_0\, B\left(\frac{(\sigma+1)N}{\sigma+2}, \frac{\sigma+1}{\sigma}+1\right)[(\sigma+2)\Gamma(N/2)]^{-1}.$$

Hence, $\|F_a\|_1$ is strictly monotone with respect to $a > 0$. Thus, for any $E_0 \in [E_-, E_+]$, equation $\|F_a\|_1 = E_0$ has a unique solution $a_0 \in [a_-, a_+]$.

With these formulae the argument of Section 4.5 can be literally repeated to supply the end of the proof of Theorem 4.10. Of course, an alternative mass analysis can be performed using the technique of Section 4.2. □

4.7 Nonexistence of fundamental solutions

Equation (4.1) with $\sigma > 0$ and $\beta \ge \beta_*$ does not admit solutions $u(x,t) \ge 0$ in Q such that $u(x,0) = 0$ for $x \ne 0$ except for the trivial one $u \equiv 0$. In particular, there exist no fundamental solutions, i.e., solutions taking on a multiple of the Dirac mass as initial data; see comments at the end of the chapter. In the critical absorption case $\beta = \beta_*$, we shall use the asymptotic description for solutions with compact support to explain what happens when we approximate the Dirac mass by a sequence of (smooth) functions with compact support. In fact, we will show that an initial layer occurs across which the solution loses its whole mass and becomes 0, the only allowed solution under those circumstances.

The connection between the behaviours for $t \to 0$ and $t \to \infty$ is based on the group of scaling transformations

$$(\mathcal{T}_\lambda u)(x,t) = \lambda^k u(\lambda^{k/N} x, \lambda t)\,, \tag{4.84}$$

which maps solutions of (4.1) into solutions of (4.1) for any $\lambda > 0$ if $k = N/(N\sigma+2)$ as above. It also preserves the total mass in the following sense:

$$\int (\mathcal{T}_\lambda u)(x,t)\,dx = \int u(x, \lambda t)\,dx\,. \tag{4.85}$$

We construct an approximation to a Dirac mass $M\delta(x)$ with $M > 0$ as follows. We take any continuous and nonnegative function with compact support $\phi(x)$ defined in $\mathbb{R}^N$ and such that $\int \phi\,dx = M$ and let for $n = 1, 2, \ldots$,

$$\phi_n(x) = n^N \phi(nx)\,. \tag{4.86}$$

Clearly ϕ_n converges to $M\delta$ in the weak topology of measures in $\mathbb{R}^N$. Now let u (resp. u_n) be the solution to (4.1), (4.2) corresponding to initial data ϕ (resp. ϕ_n). As n grows, we have

Theorem 4.12 *As $n \to \infty$, the sequence $u_n(x,t)$ converges to 0 uniformly in sets of the form $\mathbb{R}^N \times (t_0, \infty)$ for any $t_0 > 0$. Moreover, for all large $n \geq n(t_0, \phi)$, we have for $t \geq t_0$,*

$$\int u_n(x,t)\,dx \leq C\left(\ln t + \frac{N}{k}\ln n\right)^{-N/2}, \tag{4.87}$$

$$u_n(x,t) \leq Ct^{-k}\left(\ln t + \frac{N}{k}\ln n\right)^{-k}, \tag{4.88}$$

where C depends only on σ and N but not on t_0 or ϕ.

The proof consists in observing that $u_n = \mathcal{T}_{\lambda_n} u$ with $\lambda_n = n^{N/k}$, since both u_n and $\mathcal{T}_{\lambda_n} u$ are solutions of (4.1) and their initial data coincide. The estimates are then a simple consequence of Theorem 4.1 applied to u.

The same result is true and the same argument applies if we replace equation (4.1) by equation (4.14), also with critical exponent $\beta = \beta_*$. Of course, we have to change k into $N/[\sigma(N+1)+2]$.

4.8 Solutions with L^1 data

We have proved that a solution $u(x,t)$ of equation (4.1) such that its initial data $u(x,0)$ are compactly supported will eventually decay like $O((t\ln t)^{-k})$ as $t\to\infty$, while its mass will decay as $O((\ln t)^{-N/2})$. The assumption of compact support plays here an important role; as we have announced in Theorem 4.2 such estimates are not true for general initial data $u(x,0)\in L^1(\mathbb{R}^N)$, $u(x,0)\geq 0$. The present section will be devoted to proving this fact.

Before proceeding with the proof of Theorem 4.2, we observe that, as a subsolution to the PME $u_t=\Delta u^{\sigma+1}$, a solution of (4.1) will have a nonincreasing mass function $M(t)$ and will also satisfy the following uniform decay rate:

$$u(x,t)\leq C(\sigma,N)(M(0))^{2k/N}t^{-k} \tag{4.89}$$

with $k=N/(N\sigma+2)$ as before. Nothing better than these quantitative estimates can be obtained for any finite time interval if the only information we have on the initial data is the mass $M(0)<\infty$, and this applies even for smooth data with compact support.

Lemma 4.13 *Given a certain time $T>0$ and a fixed initial mass $M>0$, there exist solutions of* (4.1) *with compactly supported initial data $u(x,0)$ and $\int u(x,0)\,dx=M$, such that the following holds. For every $0\leq t\leq T$*

$$M(t)>\frac{M}{2}\quad \textit{and}\quad \|u(\cdot,t)\|_\infty\geq c(t+T)^{-k}, \tag{4.90}$$

where $c>0$ does not depend on T.

Proof. Let us pick any smooth initial data $v_0\in L^1(\mathbb{R}^N)$ such that $v_0(x)\geq 0$, $\int v_0\,dx=M$ and v_0 is positive at $x=0$. Solving problem (4.1), (4.2) with initial data v_0 produces a function $v\in C([0,\infty):L^1(\mathbb{R}^N))$. Hence, there exists $t_0>0$ such that $\int v(x,t)\,dx>M/2$ for $0\leq t\leq t_0$. Let

$$c=\inf\{v(0,t)(t+t_0)^k : 0\leq t\leq t_0\}>0\,.$$

We now apply to v the scaling transformation (4.84) and put $u=\mathcal{T}_\lambda v$ with $\lambda=t_0/T$. Since the transformation is mass preserving, we get $\int u(x,0)\,dx=M$ and

$$\int u(x,t)\,dx>\frac{M}{2} \tag{4.91}$$

for $0\leq t\leq t_0/\lambda=T$. Moreover, we have in this interval

$$(t+T)^k\|u(\cdot,t)\|_\infty=\lambda^k(t+T)^k\|v(\cdot,\lambda t)\|_\infty\geq c\,. \tag{4.92}$$

□

The preceding proof is based on the properties of the group of scaling transformations $\mathcal{T}_\lambda$ which for $\lambda<1$ flattens the data. If we eliminate the restriction of compact support, the same effect will allow us below to construct solutions whose decay rates have no logarithmic factors.

Proof of Theorem 4.2. Part (1). We begin by establishing both decay rates. The mass estimate follows from Theorem 4.1 by approximation. Thus, given $\varepsilon > 0$, we may find a function $v_0 \in C(\mathbb{R}^N)$, $v_0 \geq 0$, with compact support and such that $\|u_0 - v_0\|_1 \leq \varepsilon/2$. From standard properties, it then follows that for every $t > 0$,

$$\|u(t) - v(t)\|_1 \leq \tfrac{\varepsilon}{2}, \tag{4.93}$$

where $v(t) = v(\cdot, t)$ is the solution of (4.1) with initial data v_0. Since by Theorem 4.1 $\int v(t)\, dx \to 0$ as $t \to \infty$, there exists $T > 0$ such that for $t > T$,

$$\int u(t)\, dx \leq \|u(t) - v(t)\|_1 + \int v(t)\, dx \leq \varepsilon,$$

which settles (4.16). As for the L^∞-estimate (4.17), we only have to recall that u is a subsolution to the PME (4.3) and use estimate (4.89) with origin of time at $t_1 < t$ to obtain

$$u(x,t) \leq C(M(t_1))^{2k/N}(t - t_1)^{-k},$$

from which (4.17) immediately follows (let $t_1 = t/2$ for instance).

Thanks to the maximum principle, the above estimates are also true for solutions of changing sign. The equation has to be changed accordingly into

$$u_t = \Delta(|u|^\sigma u) - |u|^{\beta-1}u. \tag{4.94}$$

We leave the details to the reader.

Part (2). We now show that (4.16), (4.17) cannot be improved. We consider an arbitrary decay rate given by a continuous decreasing function $g(t) > 0$ defined for $t > 0$ and such that $g(t) \to 0$ as $t \to \infty$ and construct a solution u of (4.1) with prescribed mass $M > 0$ such that for an infinite sequence of times $\{t_n\} \to \infty$,

$$M(t_n) = \int u(t_n)\, dx \geq g(t_n) \quad \text{and} \quad u(x_n, t_n)t_n^k \geq g(t_n) \tag{4.95}$$

for some $x_n \in \mathbb{R}^N$. The proof consists in suitably transforming a solution v with smooth and compactly supported initial data, like the one considered in Lemma 4.13, by means of the group of scalings $\mathcal{T}_\lambda$ described above and $\mathcal{S}_\mu$ defined by

$$(\mathcal{S}_\mu v)(x,t) = \mu v(\mu^{-\sigma/2}x, t), \tag{4.96}$$

which for $\mu \in (0,1)$ transforms a *solution* of (4.1) with initial mass $M > 0$ into a *subsolution* of the same equation with mass $\mu^{N/2k}M$. Let then

$$u_n(x,t) = (\mathcal{S}_{\mu_n}\mathcal{T}_{\lambda_n}v)(x,t) \tag{4.97}$$

and assume that:

(i) $\int v(x,0)\,dx = M/2$,
(ii) $\mu_n = 2^{-2kn/N}$, and
(iii) the λ_n's form a decreasing sequence determined as follows. There exists $\tau > 0$ such that $\int v(t)\,dx > M/4$ for $0 \le t \le \tau$. Since $g(t) \to 0$ as $t \to \infty$, there exists $t_n \ge n$ such that

$$g(t_n) < \min\{(1/2)^{n+2}M,\ \tau^k 2^{-2kn/N}\|v(\tau)\|_\infty\}\,. \tag{4.98}$$

Then we set $\lambda_n = \tau/t_n$.

With these definitions we obtain the following mass estimate for u_n:

$$\int u_n(x,t_n)\,dx = \mu_n^{N/2k}\int v(x,\lambda_n t)\,dx \ge 2^{-n}M/4 \ge g(t_n) \tag{4.99}$$

if $0 \le t \le t_n$. We also have

$$t_n^k\|u_n(\cdot,t_n)\|_\infty = \mu_n\lambda_n^k t_n^k\|v(\cdot,\lambda_n t_n)\|_\infty \ge g(t_n). \tag{4.100}$$

Finally, we define u as the solution of (4.1) with initial data

$$u(x,0) = \sum_n u_n(x,0)\,. \tag{4.101}$$

We observe that $\int u(x,0)\,dx = M$. On the other hand, since u_n is a subsolution of the same equation and $u(x,0) \ge u_n(x,0)$, we conclude by the standard comparison that $u(x,t) \ge u_n(x,t)$ for every $x \in \mathbb{R}^N$ and $t > 0$. Together with (4.99) and (4.100) this implies the desired estimates (4.95). □

The same arguments apply literally to equation (4.14). Besides, there is no major difficulty in applying them to the semilinear case $\sigma = 0$, as the reader may easily check.

4.9 General nonlinearity

The above results apply to equations with power-like nonlinearities which are invariant under a group of scaling transformations. Though use of this group was essential in the proofs, these can be adapted to a number of equations which can be viewed as *small perturbations* of the above ones. In order to show how to proceed in those cases we will consider briefly the asymptotic behaviour of the solutions to the equation

$$u_t = \Delta(u^{\sigma+1}) - f(u) \quad \text{in } Q\,, \tag{4.102}$$

where $\sigma \ge 0$ and $f : [0,\infty) \to [0,\infty)$ is a continuous function such that $f(0) = 0$. We also assume that

$$u(x,0) = u_0(x) \quad \text{for } x \in \mathbb{R}^N \tag{4.103}$$

with $u_0 \not\equiv 0$, integrable, nonnegative and compactly supported. Then there exists a unique solution $u \in C([0,\infty) : L^1(\mathbb{R}^N))$ of problem (4.102), (4.103). Moreover, since $f \geq 0$ it follows from the maximum principle that such a solution is bounded above by the solution of the purely PME (4.3) with same initial data, which means that estimate (4.89) holds and therefore u will decay as $t \to \infty$ with at least the rate $O(t^{-k})$. In order to obtain more precise estimates for the solution let us make the following assumption of *critical growth* on the absorption term:

$$f(s) \geq C_1 \, s^{\beta_*} \quad \text{for } 0 \leq s \leq c\,, \tag{4.104}$$

for some $C_1, \; c > 0$. We first remark that, by estimate (4.89), u will be bounded above by c. It is then clear under these assumptions that whenever $u(x,t)$ is a solution of (4.1), then the function

$$v(x,t) = u(\sqrt{C_1}x, C_1 t) \tag{4.105}$$

will be a supersolution to the equation (4.1) once t is large enough. Hence, by the maximum principle, we obtain a decay rate for u of the form $O((t \ln t)^{-k})$ as $t \to \infty$. More precisely, as a consequence of the estimates of Theorem 4.1, for every $\varepsilon > 0$ we can find $t_\varepsilon > 0$ such that whenever $t \geq t_\varepsilon$,

$$\int u(x,t)\,dx \leq (C_* + \varepsilon)(\ln(C_1 t))^{-N/2}\,, \tag{4.106}$$

$$u(x,t) \leq (C_1 t \ln(C_1 t))^{-k} \left(F_{a_*}(\xi) + \varepsilon\right)\,, \tag{4.107}$$

where $\xi = x\,t^{-k/N}[\ln(C_1 t)]^{k\sigma/2} C_1^{-k\sigma/2}$ and C_* and a_* are as in Theorem 4.1. Comparison from below under the assumption

$$f(s) \leq C_2 \, s^{\beta_*} \quad \text{for } 0 \leq s \leq c \tag{4.108}$$

can be done in a similar way. Putting together both results, we obtain the following

Theorem 4.14 *Let u be a solution of problem* (4.102), (4.103) *under the assumptions specified above, and assume that moreover*

$$\lim_{s \to 0} f(s)/s^{\beta_*} = 1\,. \tag{4.109}$$

Then Theorem 4.1 *holds.*

If the above limit equals $C > 0$, $C \neq 1$ instead of 1, we apply Theorem 4.14 to the rescaled function $u(\sqrt{C}x, Ct)$.

It is evident that the same results apply if a perturbation of the absorption term is considered for equation (4.14). Other examples will need more work. Take for instance the case of the equation $u_t = \Delta\phi(u) - f(u)$, where ϕ is a small perturbation of the power $u^{\sigma+1}$ and f is as above. We leave the precise details an an exercise for the interested reader.

4.10 Dipole-like behaviour with critical absorption exponents in a half line and related problems

Statement of the problems and discussion. In this section we study the asymptotic behaviour as $t \to \infty$ of the solution of the initial boundary value problem for the quasilinear heat equation (PME) with absorption

$$u_t = (u^m)_{xx} - u^p \quad \text{in } Q = \mathbb{R}_+ \times \mathbb{R}_+ , \tag{4.110}$$

$$u(0,t) = 0 \quad \text{for } t \geq 0 , \tag{4.111}$$

$$u(x,0) = u_0(x) \quad \text{for } x > 0, \tag{4.112}$$

where $m > 1$ and $p > m$ are constants. For convenience, we use the standard PME form of the diffusion operator, so that this corresponds to $\sigma = m - 1$ in the previous sections. We assume that the initial datum $u_0 \not\equiv 0$ is given by an integrable, nonnegative and compactly supported function. The semilinear case, $m = 1$, will also be considered. This analysis is motivated by the results of previous sections where the Cauchy problem for the N-dimensional heat equation with absorption was considered for the critical exponent $p = m + 2/N$. The main conclusion of such analysis can be summarized as saying that there are several exponent ranges corresponding to different ways in which diffusion and absorption interact for large times, and that in each of these ranges a simple asymptotic description can be given in terms of special solutions.

In view of this clear picture, it is natural to investigate to what extent it reflects a general phenomenon, valid for different equations and/or settings. Our purpose here is to address the question of the influence of the boundary conditions. Condition (4.111) means an extra heat extraction at the endpoint $x = 0$, so that in particular the decay as $t \to \infty$ is expected to be faster and the expansion slower (in fact, including the extra extraction process at $x = 0$ makes the problem more difficult than the Cauchy one and, as we will see, computations become more involved). We will show for this problem that a similar classification holds, though with different parameter regions, decay and expansion rates, and also different asymptotic profiles, which are self-similar solutions of *dipole* type.

The main points of the present classification are as follows:

(i) There exist three different situations according to the relative values of m and p, namely when p equals $p_* = m+1$ (critical absorption exponent), the supercritical case for larger p and the subcritical case for smaller p.

(ii) In the supercritical case the absorptive effects are negligible for large times, while in the subcritical case (precisely for $m < p < p_*$) both effects enter into the equation satisfied by the asymptotic profile and in determining the rates of convergence. Finally, in the critical case a subtle limit situation happens, and, while the shape of the limit profile is purely diffusive, the rates of convergence are affected by logarithmic corrections due to the influence of the absorption term.

(iii) In all cases the family of asymptotic profiles is identified and convergence with suitable rates is established. In particular, in the subcritical case we construct a new (a unique) self-similar solution of (4.110), (4.111) which is stable in the evolution, and moreover is the universal attractor of all the solutions of the problem (with finite momentum, see details below). This is a *very singular solution* (VSS), i.e., as $t \to 0$, it concentrates at $x = 0$ with infinite initial momentum.

The analysis of the Cauchy problem with nonnegative data in the critical case done in the previous sections led to a classification similar in outline, with critical exponent $p_* = m + 2/N$. In fact, the present results can be considered as a further step in the study of asymptotic behaviour for the Cauchy problem in the whole line, $x \in \mathbb{R}$. Thus, if we extend the solution of (4.110)–(4.112) to negative x by putting $u(-x, t) = -u(x, t)$ (using the obvious reflection invariance $x \mapsto -x$ of the equation), we obtain a solution of the degenerate parabolic equation

$$u_t = (|u|^{m-1}u)_{xx} - |u|^{p-1}u \tag{4.113}$$

with antisymmetric data $u_0(-x) = -u_0(x)$ such that $u_0(x) \geq 0$ for $x \geq 0$. This is the simplest class of solutions with changing sign whose behaviour departs from the results above. In the nonabsorption case (see references in the comments at the end of the chapter) the *dipole solutions* (i.e., solutions with initial data $c\delta'(x)$) explain the asymptotic behaviour of the Cauchy problem for the *signed porous medium equation*,

$$u_t = (|u|^{m-1}u)_{xx} \tag{4.114}$$

for all initial data which satisfy $\int u_0(x)dx = 0$ and $\int xu_0(x)dx \neq 0$. Such special solutions are used in the present analysis to explain the asymptotic behaviour in the critical and supercritical cases.

Finally, let us remark that our analysis not only gives the rates of decay and the asymptotic profiles of the solutions as $t \to \infty$, but also allows us to estimate the behaviour of the interfaces which bound the support of the solution. Details on this subject are given in the concluding subsection. The asymptotic results we have just described extend to equations with more general absorption terms, of the form

$$u_t = (u^m)_{xx} - f(u), \tag{4.115}$$

where $f(u)$ behaves like the power function Cu^p for $u \approx 0$. Indeed, only what happens for $u \approx 0$ matters. In that sense we can also consider more general diffusion nonlinearities

$$u_t = (\phi(u))_{xx} - f(u), \tag{4.116}$$

again under the condition that $\phi(u)$ behaves like a power. The main estimates will again be the same as in the pure-power case, though the proofs will not be so immediate.

Statement of results. Since the *critical case* $p = m + 1$ seems to be the most interesting one mathematically, we will devote most of our effort to it. In order to find the asymptotic profile we observe that, as is well known, the purely diffusive equation

$$u_t = (u^m)_{xx} \quad \text{in } Q \tag{4.117}$$

admits the one-parameter family of explicit dipole-type solutions of the form

$$\mathcal{U}_M(x,t) = t^{-1/m} F(\xi; c), \quad \xi = x/t^{1/2m}, \tag{4.118}$$

satisfying (4.111) and the law of *momentum conservation*:

$$\int_0^\infty x\,\mathcal{U}_M(x,t)\,dx = M > 0 \quad \text{for } t > 0. \tag{4.119}$$

By substituting (4.118) into equation (4.117), one can see that $F(\xi; c) \equiv F_c(\xi)$ satisfies

$$\mathbf{A}(F) \equiv (F^m)'' + \frac{1}{2m}\xi F' + \frac{1}{m} F = 0 \quad \text{for } \xi > 0, \tag{4.120}$$

and $F(0; c) = 0$. Notice that $\xi \mathbf{A}(F) \equiv \xi (F^m)'' + (F\xi^2)'/2m$. Integrating (4.120), we deduce the formula for the *dipole profile*

$$F(\xi; c) = A_0 \xi^{1/m} \big[c^{(m+1)/m} - \xi^{(m+1)/m}\big]_+^{1/(m-1)} \quad \text{for } \xi \ge 0, \tag{4.121}$$

where $A_0 = [(m-1)/2m(m+1)]^{1/(m-1)} > 0$. The free parameter $c > 0$ in (4.121) can be determined as a function of M using (4.119):

$$c = \kappa(m)\, M^{(m-1)/2m}. \tag{4.122}$$

As a curiosity, κ is given by the expression

$$\kappa(m) = \left\{ (m+1) \left[m A_0 B \left(\frac{2m+1}{m+1}, \frac{m}{m-1} \right) \right]^{-1} \right\}^{(m-1)/2m}$$

(B is Euler's Beta function). This dipole solution was first constructed by Barenblatt and Zel'dovich in 1957 [28]. Recall that when extended antisymmetrically to the whole line $x \in \mathbb{R}$, it solves the signed PME (4.114) with initial data $-M\,\delta'(x)$.

We now state the main result covering the behaviour for a critical absorption exponent. It is convenient to introduce the rescaled function

$$\theta(\xi,\tau) = [(2+t)\ln(2+t)]^{1/m} u(\xi(2+t)^{1/2m}[\ln(2+t)]^{-(m-1)/2m}, t), \tag{4.123}$$

and the new time variable is defined as $\tau = \ln(2+t)$ (the 2 plays no role and we use it for convenience to have $\tau(0) > 0$). But for the logarithmic factors, the rescaling is inspired by the dipole solution (4.118) of the PME. The occurrence of these new logarithmic factors represents the effect of the absorption term. We have:

Theorem 4.15 *Let $m > 1$, $p = m+1$ and let u_0 satisfy the above hypotheses. Then, as $\tau \to \infty$,*

$$\theta(\xi,\tau) \to F(\xi; c_*) \tag{4.124}$$

uniformly in $\xi \in \mathbb{R}_+$. Here $c_* > 0$ is a universal constant which depends only on m.

As in the critical behaviour for the Cauchy problem, we find that there exists a single asymptotic profile, and no trace of the initial data (taken in the given class, of course) is preserved in the limit $t \to \infty$. For the record, the exact value of c_* is

$$c_* = \left[A_0^{-m} B\left(\frac{2m+1}{m+1}, \frac{m}{m-1}\right) / B\left(\frac{3m+1}{m+1}, \frac{2m}{m-1}\right)\right]^{(m-1)/2m} . \tag{4.125}$$

As to the proof of this result, let us first observe that θ satisfies the equation

$$\theta_\tau = \mathbf{A}(\theta) + \frac{1}{\tau}\mathbf{C}(\theta), \qquad \xi > 0,\ \tau > \tau_0 = \ln 2, \tag{4.126}$$

with Dirichlet boundary condition at $\xi = 0$,

$$\theta(0, \tau) = 0, \qquad \tau \geq \tau_0, \tag{4.127}$$

and initial data at $\tau = \tau_0$,

$$\theta(\xi, \tau_0) = \theta_0(\xi) \equiv (2\ln 2)^{1/m} u_0(\xi 2^{1/2m} (\ln 2)^{-(m-1)/2m}), \quad \xi > 0. \tag{4.128}$$

In (4.126), $\mathbf{A}(\theta)$ is the autonomous operator given by (4.120) and

$$\mathbf{C}(\theta) = \frac{1}{m}\theta - \frac{m-1}{2m}\xi\theta' - \theta^{m+1}. \tag{4.129}$$

The study of the convergence (4.124) begins with some explicit lower and upper estimates of the solution to (4.110)–(4.112) obtained via the construction of suitable super- and subsolutions. Next we use the S-Theorem from Chapter 1. The term $\tau^{-1}\mathbf{C}(\theta)$ plays here the role of a small perturbation and then the remaining equation $\theta_\tau = \mathbf{A}(\theta)$ is just the rescaled PME.

Since the function $F(\xi; c)$ solves the stationary equation (4.120), we have the problem of stabilization as $\tau \to \infty$ to a stationary solution. The uniqueness of the actual stationary solution taken in the limit, i.e., the fact that only the precise constant $c = c_*$ appears in the limit (4.124), is proved by using the rescaled *momentum equation*

$$\frac{dM(\theta(\tau))}{d\tau} = \frac{1}{\tau}\left(M(\theta(\tau)) - \langle \xi, \theta^{m+1}(\tau)\rangle\right) \tag{4.130}$$

for the first momentum of the solution

$$M(\theta(\tau)) = \langle \xi, \theta(\xi, \tau)\rangle \equiv \int_0^\infty \xi\theta(\xi, \tau) d\xi, \quad \tau > \tau_0. \tag{4.131}$$

Since the function $1/\tau$ in (4.130) is not integrable at infinity, the uniqueness of the constant c_* given by (4.125) is a straightforward consequence of the algebraic equation

$$M(F_c) - \langle \xi, F_c^{m+1}\rangle = 0\,, \tag{4.132}$$

which implies that $c > 0$ can take only the value c_*. The use of the momentum equation is the main novelty with respect to the analysis of the corresponding mass equation in the Cauchy problem.

For the semilinear heat equation with the critical absorption parameter,

$$u_t = u_{xx} - u^2 \quad \text{in } Q \tag{4.133}$$

and conditions (4.111), (4.112), the corresponding asymptotic behaviour can be stated as follows.

Theorem 4.16 *Let $m = 1$, $p = 2$ and let u_0 be some nonnegative integrable initial data such that for some constant $\gamma > 0$,*

$$u_0(x) = o(e^{-\gamma x^2}) \quad \text{as } x \to \infty. \tag{4.134}$$

Then, as $\tau \to \infty$ the rescaled function (4.123) *with $m = 1$ satisfies*

$$\theta(\xi, \tau) \to F(\xi; c_*) \equiv c_* \xi e^{-\xi^2/4} \tag{4.135}$$

uniformly in ξ, where $c_* = \pi^{1/2}$.

The asymptotic analysis is quite similar. One can see that $\theta(\xi, \tau)$ solves the same equation (4.126), where the operators $\mathbf{A}(\theta)$ and $\mathbf{C}(\theta)$ are given in (4.120) and (4.129) with $m = 1$. Since $\mathbf{A}$ is linear, the limiting equation $\theta_\tau = \mathbf{A}(\theta)$ admits a linear family of stationary solutions $\{F(\xi; c) = c\xi e^{-\xi^2/4},\ c \geq 0\}$. The unique choice of $c = c_*$, which occurs in the limit, follows from (4.132) for $m = 1$, and sharp estimates of the solution.

Once the critical case $p = m + 1$ is settled, we study the cases $m < p < m + 1$ and $p > m+1$. This shows that the exponent $p = m+1$ is really *critical* in problem (4.110)–(4.112).

For the *subcritical* case $m < p < m + 1$ the behaviour as $t \to \infty$ is described by a self-similar solution of equation (4.110) of the form

$$V(x,t) = t^{-1/(p-1)} f(\xi), \quad \xi = x/t^{(p-m)/2(p-1)}, \tag{4.136}$$

i.e., both absorption and diffusion are involved in the asymptotic behaviour. The profile f satisfies the ordinary differential equation

$$(f^m)'' + \frac{p-m}{2(p-1)} f'\xi + \frac{1}{p-1} f - f^p = 0 \quad \text{for } \xi > 0, \tag{4.137}$$

$$f(0) = 0, \quad f(\xi) > 0 \quad \text{for small } \xi > 0. \tag{4.138}$$

The following result holds.

Theorem 4.17 *Let $p \in (m, m+1)$. Then:*

(i) There exists a unique compactly supported nontrivial solution $f \geq 0$ of the ordinary differential problem (4.137), (4.138).

(ii) Under the given hypotheses on u_0, as $\tau = \ln(1+t) \to \infty$ uniformly in $\mathbb{R}_+$

$$\theta(\xi, \tau) \equiv (1+t)^{1/(p-1)} u(\xi(1+t)^{(p-m)/2(p-1)}, t) \to f(\xi). \tag{4.139}$$

We remark that the momentum evolution of the special solution (4.136) is given by the formula

$$\int x V(x,t)\, dx = c_0 t^{-\mu} \quad \text{with } \mu = (m+1-p)/(p-1) > 0,$$

so that V is given the name of *very singular self-similar solution*, because it starts at $t = 0$ with an infinite momentum. The term "very singular" refers to the divergence as $t \to 0$ of the momentum (or mass in the Cauchy problem). Both types of very singular solution are different: the one constructed here, when viewed as a function defined for $x \in \mathbb{R}$, is antisymmetric, while the VSS of the Cauchy problem is symmetric. The former has infinite initial momentum, the latter infinite mass. Both are the unique profiles to which solutions of the respective problems converge. Observe finally that the rescaling exponents are the same, since they are uniquely determined by the equation, though the range of application is different.

In the *supercritical* case $p > m+1$ the asymptotic behaviour is quite simple: the effect of the absorption term is negligible for large times and the structure of the solution as $t \to \infty$ is described by the explicit dipole solution (4.118), (4.121) of the PME. The influence of absorption as $t \to \infty$ is reflected only in the final constant M_∞ (the momentum of the solution $u(x,t)$ at $t = \infty$).

Theorem 4.18 *Let $p > m+1$. Then, under the given hypotheses on u_0, there exists a constant $c_\infty > 0$ depending on u_0 such that as $\tau \to \infty$,*

$$\theta(\xi, \tau) \equiv (1+t)^{1/m} u(\xi(1+t)^{1/2m}, t) \to F(\xi; c_\infty) \tag{4.140}$$

uniformly in $\mathbb{R}_+$, where F is the dipole profile (4.121).

In comparing this result with Theorem 4.15, we observe that there are no extra logarithmic factors, or other, superposed to the rescaling (4.118). We also remark that c_∞ is not a universal constant, but may range, depending on the initial data, over the whole of $\mathbb{R}_+$.

Lower and upper bounds in the critical case. Quasilinear case $m > 1$. The first step in the proof of Theorem 4.15, corresponding to the critical absorption exponent $p = m+1$ in equation (4.110), is to establish explicit estimates from below and above. These can be done by the construction of suitable weak sub- and supersolutions to equation (4.110). These constructions are made specially for the case $p = m+1$. The constructions for the other cases will be done later on.

We begin with the lower bound. It is expressed in terms of the function $F(\xi; c)$ defined in (4.121).

Lemma 4.19 *If $c_- > 0$ is small enough and $T > 1$ is large, we have*

$$u(x,t) \geq [(T+t)\ln(T+t)]^{-1/m} F(\xi; c_-) \tag{4.141}$$

for all $x \in \mathbb{R}_+$ and $t > T$. Here the rescaled space variable is given in accordance with (4.123) *by*

$$\xi = x(T+t)^{-1/2m}[\ln(T+t)]^{(m-1)/2m}. \tag{4.142}$$

Proof. Let us define

$$u_-(x,t) = [(T+t)\ln(T+t)]^{-1/m} F(\xi; c_-). \tag{4.143}$$

By definition, a smooth enough function like u_- will be a weak subsolution to equation (4.110) if the corresponding rescaled function

$$\theta_-(\xi,\tau) = [(T+t)\ln(T+t)]^{\frac{1}{m}} u_-(\xi(T+t)^{\frac{1}{2m}}[\ln(T+t)]^{-\frac{m-1}{2m}}, t) \tag{4.144}$$

satisfies inside its domain of positivity the inequality

$$(\theta_-)_\tau \leq \mathbf{A}(\theta_-) + \frac{1}{\tau}\mathbf{C}(\theta_-) \tag{4.145}$$

for $\xi > 0$ and $\tau = \ln(T+t) > \tau_* = \ln(2T)$. Since $\theta_- \equiv F(\xi; c_-)$ and hence $\mathbf{A}(\theta_-) \equiv 0$ and $(\theta_-)_\tau \equiv 0$, (4.145) is valid if

$$\mathbf{C}(F(\xi; c_-)) \geq 0 \quad \text{for} \quad 0 < \xi < c_-. \tag{4.146}$$

Writing $D = c_-^{(m+1)/m} - \xi^{(m+1)/m}$, this is equivalent to the inequality

$$\frac{m+1}{2m^2} + \frac{m+1}{2m^2}\xi^{(m+1)/m} D^{-1} - A_0^m \xi D^{m/(m-1)} \geq 0, \quad 0 < \xi < c_-. \tag{4.147}$$

It is now clear that (4.146) is valid for any $c_- > 0$ small enough.

On the other hand, by well-known properties of weak solutions to the problem (4.110)–(4.112), for any $T > 1$ large enough, $u(x,T) > 0$ in a small right-hand neighbourhood of the point $x = 0$. Moreover, we can suppose that

$$(u^m)_x(0,T) > 0. \tag{4.148}$$

Indeed, if $(u^m)_x(0,T) = 0$ for any large $T > 1$, then the function $u(x,t)$ becomes the solution of the Cauchy problem for equation (4.110) in $\mathbb{R}_+ \times (0,\infty)$ and it is localized in x from the left. This contradicts known properties of the solution to the Cauchy problem. Thus, by choosing $T > 1$ so that (4.148) holds and $c_- > 0$ small enough, we deduce that $u_-(x,T) \leq u(x,T)$ for $x \in \mathbb{R}_+$. By comparison we arrive at estimate (4.141). □

The construction of the supersolution is rather complicated. Notice that a supersolution of a simple form similar to that given in the right-hand side of (4.141) does not exist. We now perform another construction.

Lemma 4.20 *For any large $T > 1$, there exist constants $c_+ > 0$, $d > 0$ and $b > (m^2 - 1)/2m$ such that for all $x > 0$ and $t > T$,*

$$u(x,t) \le u_+(x,t) \equiv [(T+t)\ln(T+t)]^{-1/m} F(\eta; c_+), \tag{4.149}$$

$$\eta = [\xi + d/\ln(T+t)]\,[1 + b/\ln(T+t)]^{-m/(m+1)},$$

and ξ is as given in (4.142).

Proof. First, we need to prove that under given hypotheses, there exists a weak supersolution of the form

$$\theta_+(\xi,\tau) = F((\xi + d/\tau)(1+b/\tau)^{-m/(m+1)}; c_+) \tag{4.150}$$

satisfying the following inequality within the positivity domain (cf. (4.145)):

$$(\theta_+)_\tau \ge \mathbf{A}(\theta_+) + \frac{1}{\tau}\mathbf{C}(\theta_+) \quad \text{for} \quad \xi > 0, \tau > \tau_*. \tag{4.151}$$

By substituting (4.150) into (4.151), we have to verify the inequality (cf. 4.147))

$$m(m+1)^{-1} b\tau^{-2}(1+b/\tau)^{-1}(\xi + d/\tau) - d\tau^{-2} + (\xi + d/\tau)^{(m+1)/m} D^{-1} I_1$$

$$+ I_2 + m\tau^{-1} A_0^m (1+b/\tau)^{-m/(m+1)} (\xi + d/\tau)^2 D^{m/(m-1)} \ge 0 \tag{4.152}$$

for any $\xi \ge 0$ such that

$$D = D(\xi) \equiv c_+^{(m+1)/m} - (\xi + d/\tau)^{(m+1)/m}(1+b/\tau)^{-1} > 0.$$

We denote by I_1 and I_2 the functions

$$I_1 = -m(m-1)^{-1} b\tau^{-2}(1+b/\tau)^{-2}(\xi + d/\tau) + (m+1)(m-1)^{-1} d\tau^{-2}(1+b/\tau)^{-1}$$

$$- [(m+1)/(m-1)]^2 A_0^{m-1}(1+b/\tau)^{-(3m+1)/(m+1)}(\xi + d/\tau)$$

$$+ (m+1)[2m(m-1)]^{-1}[1 - (m-1)/\tau](1+b/\tau)^{-1}\xi, \tag{4.153}$$

$$I_2 = (m+1)(2m+1)(m-1)^{-1} A_0^{m-1}(1+b/\tau)^{-2m/(m+1)}(\xi + d/\tau)$$

$$- (2m)^{-1}[1 - (m-1)/\tau]\xi - (1 + 1/\tau)(\xi + d/\tau). \tag{4.154}$$

Fix $c_+ > 0$ large enough. Then it follows from (4.153) and (4.154) that in the first approximation for large $\tau > \tau_0$,

$$\begin{aligned} I_1 &= \frac{m+1}{2m(m-1)}\frac{1}{\tau}(\xi a_1 - d) + O\Big(\frac{1}{\tau^2}\Big), \\ I_2 &= \frac{1}{2m}\frac{1}{\tau}(d - a_2\xi) + O\Big(\frac{1}{\tau^2}\Big) \end{aligned} \tag{4.155}$$

uniformly in $\xi \in [0, 2c_+]$, where $a_1 = \frac{2mb}{m+1} - (m-1)$ and $a_2 = \frac{2m(2m+1)b}{m+1} + (m+1)$. Recall that $a_1 > 0$ provided that $b > (m^2-1)/2m$. Substituting (4.155) into (4.152) yields that we need to verify the following final inequality:

$$\nu_0(\xi + d/\tau)^{\frac{m+1}{m}} D^{-1} a_1(\xi - \xi_1) + a_2(2m)^{-1}(\xi_2 - \xi) + mA_0^m \xi^2 D^{\frac{m}{m-1}} > 0 \quad (4.156)$$

for any $\xi \geq 0$ such that $D > 0$ and $\tau > \tau_0$ large enough, where $\nu_0 = (m+1)/2m(m-1)$, $\xi_1 = d/a_1$ and $\xi_2 = d/a_2$. Therefore $0 < \xi_2 < \xi_1 < c_+$. Let us look at (4.156) carefully.

Denote by $W(\xi, \tau) \equiv W_1 + W_2 + W_3$ the function given in the left-hand side of (4.156). One can see by using the structure of the first term of W that for $\tau \gg 1$,

$$W(\xi, \tau) = \nu_0 c_+^{(m+1)/m} a_1(c_+ - \xi_1)\, D^{-1}(1 + o(1)) \quad (4.157)$$

as $D \to 0$ and hence $W(\xi, \tau) > 0$ near the point $\xi = c_+$ if $\tau \gg 1$ and $a_1 > 0$. The last inequality again yields the assumption $b > (m^2-1)/2m$ given in the statement of Lemma 4.20.

Consider now a small right-hand neighbourhood of the origin $\xi = 0$. The main part of $W(\xi, \tau)$ as $\tau \to \infty$ is

$$W_2(\xi, \tau) = a_2(2m)^{-1}(\xi_2 - \xi) > 0 \quad (4.158)$$

for $\xi \in [0, \delta]$, $\delta > 0$ is an arbitrary fixed constant small enough.

On the other hand, on the compact subset $[\delta, \xi_1]$, we have for $\tau \gg 1$,

$$W(\xi, \tau) \geq -a_2(2m)^{-1}|\xi_2 - \xi_1| + mA_0^m \delta^2 D^{\frac{m}{m-1}}(\xi_1) - \nu_0 \xi_1^{\frac{m+1}{m}} D^{-1}(\xi_1)|\delta - \xi_1|. \quad (4.159)$$

Therefore, we conclude that for large $\tau > \tau_0$,

$$W(\xi, \tau) > 0 \quad \text{on} \quad [\delta, \xi_1], \quad (4.160)$$

if $c_+ > 0$ is sufficiently large.

Finally, consider the sign of $W(\xi, \tau)$ for $\xi \in (\xi_1, c_+)$ and $\tau \gg 1$. Set $\xi = \alpha c_+$ where α is the new coordinate. Then $\alpha \in (\xi_1/c_+, 1)$. Using this coordinate yields that in the first approximation as $\tau \to \infty$,

$$\begin{aligned} W_1 &= \nu_0 a_1(\alpha c_+ - \xi_1)\alpha^{\frac{m+1}{m}}(1 - \alpha^{\frac{m+1}{m}})^{-1} > 0, \\ W_2 &= a_2(2m)^{-1}(\xi_2 - \alpha c_+) < 0, \\ W_3 &= mA_0^m \alpha^2 c_+^{\frac{3m-1}{m-1}}(1 - \alpha^{\frac{m+1}{m}})^{\frac{m}{m-1}} > 0 \end{aligned} \quad (4.161)$$

for $\alpha \in (\xi_1/c_+, 1)$. Consider the sum $W_1 + W_2$. By (4.161), $W_1 + W_2 > 0$ if

$$\alpha^{-(m+1)/m}(1-\alpha^{(m+1)/m}) \leq \frac{m+1}{m-1}\frac{a_1}{a_2}\frac{\alpha c_+ - \xi_1}{\alpha c_+ - \xi_2} \equiv f_1(c_+). \tag{4.162}$$

Since

$$f_1(c_+) = \frac{m+1}{m-1}\frac{a_1}{a_2}\left[1 + O\left(\frac{1}{c_+}\right)\right] \quad \text{as } c_+ \to \infty,$$

we conclude that

$$W_1 + W_2 > 0 \quad \text{for} \quad \alpha \in [\alpha_1 + \varepsilon, 1), \tag{4.163}$$

where $\varepsilon > 0$ is arbitrarily small and

$$\alpha_1 = \left[\frac{m+1}{m-1}\frac{a_1}{a_2} + 1\right]^{-m/(m+1)} < 1. \tag{4.164}$$

Consider the sum $W_2 + W_3$. Then (4.161) implies that $W_2 + W_3 > 0$ if

$$\alpha^{(m-1)/m}(1-\alpha^{(m+1)/m}) \geq \left(\frac{a_2}{2m^2}A_0^{-m}\right)^{(m-1)/m} c_+^{-2}\left(1 - \frac{\xi_2}{\alpha c_+}\right)^{(m-1)/m}$$

$$= \left(\frac{a_2}{2m^2}A_0^{-m}\right)^{(m-1)/m} c_+^{-2}\left[1 - \frac{m-1}{m}\frac{\xi_2}{\alpha c_+} + O(c_+^{-2})\right] \tag{4.165}$$

as $c_+ \to \infty$. Therefore, for $\tau \gg 1$,

$$W_2 + W_3 > 0 \quad \text{for } \alpha \in [\alpha_2(c_+), 1 - O(c_+^{-2})],$$

where as $c_+ \to \infty$,

$$\alpha_2(c_+) = O(c_+^{-2m/(m-1)}). \tag{4.166}$$

By using (4.163)–(4.166) and the fact that $\alpha_2(c_+) < \xi_1/c_+$ for $c_+ \gg 1$, we conclude that $W(\xi,\tau) > 0$ for $\xi \in [\xi_1, c_+)$ if c_+ is chosen large enough.

It follows from the structure of the supersolution given by (4.149) that if $T > 1$ and $c_+ > 0$ are sufficiently large, then $u_0(x) \leq u_+(x,0)$ for $x \geq 0$ (notice that $u_+(0,0) > 0$), hence the result by comparison. □

The following final result on upper and lower bounds of the rescaled function (4.123) is a straightforward consequence of Lemmas 4.19 and 4.20.

Lemma 4.21 *There exist constants $T > 1$ and $c_+ > c_- > 0$ such that for all $\xi > 0$ and $\tau > \tau_* = \ln(2T)$,*

$$F(\xi; c_-) \leq \theta(\xi,\tau) \leq F(\xi; c_+). \tag{4.167}$$

The semilinear case $m = 1$. As above, we now prove sharp lower and upper bounds of solutions to the semilinear equation (4.133) with the critical exponent $p = 2$.

Lemma 4.22 *For any fixed* $c_- > 0$ *small and* $T > 1$ *sufficiently large,*

$$u(x,t) \geq [(T+t)\ln(T+t)]^{-1} F(\xi; c_-), \quad \xi = x/(T+t)^{1/2}, \tag{4.168}$$

for all $x \in \mathbb{R}_+$ *and* $t > T$, *where* F *is as given by* (4.135).

Proof. It is similar to the proof of Lemma 4.19. Introducing a smooth subsolution of the form (4.143) with $m = 1$, we have to check that the corresponding rescaled function (4.144) satisfies (4.145). This yields an inequality much simpler than (4.147):

$$c_- \xi \, e^{-\xi^2/4} \leq 1 \quad \text{for all } \xi \geq 0. \tag{4.169}$$

Therefore, any positive $c_- \leq (e/2)^{1/2}$ is acceptable. One can see that by choosing $T \gg 1$, we have that $u(x,T) \geq u_-(x,T)$ for $x > 0$ provided that $c_- > 0$ is small enough, and (4.168) follows by comparison. □

As in the quasilinear case (cf. Lemma 4.20), the construction of a sharp upper bound is more difficult.

Lemma 4.23 *For any large* $T > 1$, *there exist positive constants* c_+, d *and* b *such that for all* $x > 0$ *and* $t > T$,

$$u(x,t) \leq u_+(x,t) \equiv [(T+t)\ln(T+t)]^{-1} F(\eta; c_+), \tag{4.170}$$

where $\eta = [\xi + d/\ln(T+t)]\,[1 + b/\ln(T+t)]^{-1/2}$.

Proof. We try in inequality (4.151), $m = 1$, the function (a smooth supersolution)

$$\theta_+(\xi,\tau) = F((\xi + k(\tau))\phi(\tau); c_+), \tag{4.171}$$

where $k(\tau)$ and $\phi(\tau)$ are smooth positive functions to be determined later. Then we obtain the following inequality:

$$\begin{aligned} & k'\left[1 - \tfrac{1}{2}\phi^2(\xi+k)^2\right] \\ & \quad + \tfrac{1}{2}\phi(\xi+k)^2\left[\tfrac{1}{2}\phi\xi - \tfrac{1}{2}\phi^3(\xi+k) - \phi'(\xi+k)\right] \\ & \quad + \left[(\xi+k)\left(\frac{\phi'}{\phi} + \tfrac{3}{2}\phi^2 - \frac{1}{\tau}\right) - \tfrac{3}{2}\xi - k\right] \\ & \quad + \frac{1}{\tau}c_+\phi(\xi+k)^2\, e^{-(\xi+k)^2\phi^2/4} \geq 0, \end{aligned}$$

which is valid for all $\xi > 0$ and $\tau > \tau_0 = \ln T \gg 1$. Selecting the functions $k(\tau) = d/\tau$ and $\phi(\tau) = (1 + b/\tau)^{-1/2}$, the above inequality becomes equivalent for $\tau \gg 1$ to

$$\begin{aligned} & \left[1 - \tfrac{1}{2}\left(1 + O\left(\tfrac{1}{\tau}\right)\right)\left(\xi + \tfrac{d}{\tau}\right)^2\right] O\left(\tfrac{1}{\tau^2}\right) \\ & \quad + \frac{1}{\tau}\left[\left(\xi + \frac{d}{\tau}\right)^2 \frac{b}{4}\left(\xi - \xi_2 + O\left(\tfrac{1}{\tau}\right)\right) + \left(1 + \frac{3b}{2}\right)\left(\xi_1 - \xi + O\left(\tfrac{1}{\tau}\right)\right) \right. \\ & \quad \left. + c_+\left(\xi + \frac{d}{\tau}\right)^2 e^{-\xi^2/4}\left(1 + O\left(\tfrac{1}{\tau}\right)\right)\right] \geq 0 \end{aligned}$$

for all $\xi \geq 0$, where $\xi_1 = d(3b+2)^{-1} < \xi_2 = d/b$. For large τ this can be further reduced to the inequality

$$\frac{b}{4}\xi^2(\xi - \xi_2) + \left(\frac{3b}{2}+1\right)(\xi_1 - \xi) + c_+\xi^2 e^{-\xi^2/4} \geq O\left(\frac{1+\xi^2}{\tau}\right). \tag{4.172}$$

It is easily seen that (4.172) is true for any positive d and b provided that c_+ and $\tau \gg 1$. Under hypothesis (4.134) we also have that $u_0(x) \leq u_+(x, 0)$ for $x > 0$, if $T > \max\{1, (2\gamma)^{-1}\}$ and $c_+ \gg 1$. Then we arrive at (4.170), thus completing the proof of Lemma 4.23. □

Proof of Theorem 4.15. Structure of ω-limits and rescaled momentum equation. We now use the dynamical systems approach from Chapter 1 and view equation (4.126) satisfied by θ as an asymptotically small perturbation of equation

$$\theta_s = \mathbf{A}(\theta)\,. \tag{4.173}$$

The S-Theorem says that, whenever certain three hypotheses are fulfilled, the ω-limits of trajectories $\{\theta(\tau)\}$ corresponding to the perturbed equation (4.126) are just members of the reduced ω-limit set Ω_* of the limit autonomous equation (4.173). Hence, in order to apply such result we need to check hypotheses (H1)–(H3a) from Chapter 1. It follows from Lemma 4.21 that the evolution trajectory $\{\theta(\cdot, \tau), \tau > \tau_0\}$ is uniformly bounded, and hence by known regularity results for equations of porous medium type, it is compact in $C(\mathbb{R}_+)$, and in particular

$$|(\theta^{m-1})_\xi| \leq C \quad \text{in} \quad \mathbb{R}_+ \times (\tau_1, \infty), \;\; \tau_1 = \tau_0 + 1. \tag{4.174}$$

This yields that the hypothesis (H1) is valid. Denote by $\omega(\theta_0) = \{f \in C(\mathbb{R}_+) :$ $\exists$ a sequence $\{\tau_j\} \to \infty$ such that $\theta(\cdot, \tau_j) \to f(\cdot)$ as $j \to \infty$ uniformly in $\mathbb{R}_+$, where $f \geq 0, f(0) = 0, F(\cdot; c_-) \leq f(\cdot) \leq F(\cdot; c_+)$ in $\mathbb{R}_+\}$ the ω-limit set to the problem (4.126)–(4.128). By a standard regularity result, if $\theta(\cdot, \tau_j) \to f(\cdot)$ as $\tau_j \to \infty$, then we conclude from (4.167) and (4.174) that $\theta(\cdot, \tau_j + s) \to w(\cdot, s)$ in $L^\infty_{\text{loc}}((0, \infty) : L^1(\mathbb{R}_+))$, where the function $w(\xi, s)$ solves (cf. hypothesis (H2)) equation (4.173) with data

$$w(0, s) = 0 \quad \text{for} \quad s \geq 0, \quad w(\xi, 0) = f(\xi) \quad \text{for} \quad \xi > 0. \tag{4.175}$$

Finally, we notice that the rescaled PME (4.173) with condition (4.175) generates a semigroup of contractions in the weighted momentum space $L^1_\xi(\mathbb{R}_+)$. One can prove that for every two solutions w_1 and w_2 of the problem (4.173), (4.175) with compactly supported nonnegative initial data $w_1(\cdot, 0)$ and $w_2(\cdot, 0)$ from $L^1_\xi(\mathbb{R}_+)$ and every $s > 0$,

$$\int_0^\infty \xi[w_1(\xi, s) - w_2(\xi, s)]_+ d\xi \leq \int_0^\infty \xi[w_1(\xi, 0) - w_2(\xi, 0)]_+ d\xi. \tag{4.176}$$

This implies that under above hypotheses the reduced ω-limit set Ω_* of the problem (4.173), (4.175),

$$\Omega_* = \{F(\xi; c) : c_- \le c \le c_+\}, \tag{4.177}$$

is uniformly Lyapunov stable in $L^1_\xi(\mathbb{R}_+)$, cf. hypothesis (H3a).

Thus, by the S-Theorem we conclude that

$$\omega(\theta_0) \subseteq \Omega_*. \tag{4.178}$$

The unique choice of the parameter $c = c_* > 0$ solving the algebraic equation (4.132) is proved by using the momentum equation (4.130) as in Section 4.5. The analysis of the semilinear problem for equation (4.133) and the proof of Theorem 4.16 offers no novelties.

Subcritical case $p < m + 1$. Proof of Theorem 4.17. The rescaled function given in (4.139) solves the following quasilinear equation:

$$\theta_\tau = \mathbf{B}(\theta) \equiv (\theta^m)_{\xi\xi} + \frac{p-m}{2(p-1)}\theta'\xi + \frac{1}{p-1}\theta - \theta^p \quad \text{in } Q, \tag{4.179}$$

$$\theta(0, \tau) = 0, \quad \tau \ge 0; \quad \theta(\xi, 0) = \theta_0(\xi) \equiv u_0(\xi), \quad \xi > 0. \tag{4.180}$$

The proof of Theorem 4.17 consists of several steps. We begin by the construction of upper and lower bounds.

Lemma 4.24 *Let $p \in (m, m+1)$. Then there exist positive constants $c_\pm, \alpha_\pm$ such that as $\tau \to \infty$, for all $\xi \ge 0$, there holds*

$$\alpha_- F(\xi; c_-) \le \theta(\xi, \tau) \le \alpha_+ F(\xi; c_+). \tag{4.181}$$

Proof. Weak subsolution. First, we shall look for a weak subsolution to equation (4.179) of the form

$$\theta_-(\xi) = \alpha_- F(\xi; c_-). \tag{4.182}$$

By using the fact that F solves (4.120), we deduce that $\mathbf{B}(\theta) \ge 0$ a.e. in $\mathbb{R}_+$ if

$$\frac{m+1}{m-1} D^{-1} \xi^{(m+1)/m} \left[\frac{1}{2m}\alpha_-^{m-1} - \frac{p-m}{2(p-1)}\right] + \left[\frac{p+m}{2(p-1)} - \frac{2m+1}{2m}\alpha_-^{m-1}\right] - m\,(\alpha_- A_0)^{p-1}\,\xi^{(p-1)/m} D^{(p-1)/(m-1)} \ge 0$$

for $\xi \in (0, c_-)$, where $D = c_-^{(m+1)/m} - \xi^{(m+1)/m}$. One can see from the first two terms of this inequality that the following conditions have to be valid:

$$\frac{1}{2m}\alpha_-^{m-1} - \frac{p-m}{2(p-1)} \ge 0, \quad \frac{p+m}{2(p-1)} - \frac{2m+1}{2m}\alpha_-^{m-1} > 0. \tag{4.183}$$

Hence α_- must satisfy

$$\frac{m(p-m)}{p-1} \le \alpha_-^{m-1} < \frac{m(p+m)}{(p-1)(2m+1)}. \tag{4.184}$$

Such an $\alpha_- > 0$ exists if $p < m+1$. Fix now an α_- satisfying (4.184) and set

$$B = \frac{p+m}{2(p-1)} - \frac{2m+1}{2m}\alpha_-^{m-1} > 0. \tag{4.185}$$

It follows that the function (4.182) is a subsolution if

$$B - m\,(\alpha_- A_0]^{p-1} \sup_{\xi\in(0,c_-)} \xi^{\frac{p-1}{m}} D^{\frac{p-1}{m-1}}$$
$$\equiv B - m\,(\alpha_- A_0)^{p-1}\left(\frac{m-1}{2m}\right)^{\frac{p-1}{m+1}}\left(\frac{m+1}{2m}\right)^{\frac{p-1}{m-1}} c_-^{\frac{2(p-1)}{m-1}} \ge 0, \tag{4.186}$$

i.e., if $c_- > 0$ is small enough. Finally, using a technique similar to that given in the proof of Lemma 4.19, we obtain that there exists $\tau_2 > 0$ large enough such that for small $c_- > 0$, there holds $\theta(\xi, \tau_2) \ge \theta_-(\xi)$ for $\xi \ge 0$. Hence, by comparison we obtain the left-hand inequality in (4.181).

Weak supersolution. First of all, the reader may easily convince himself that a supersolution to equation (4.179) of the simplest form given by (4.182) does not exist. Instead we try the following function (a weak supersolution):

$$\theta_+(\xi) = A\xi^{1/m}\left(c^\gamma - \xi^\gamma\right)_+^{1/(m-1)}, \tag{4.187}$$

where $A > 0$, $c > 0$ and $\gamma \in (0, 1+1/m)$ are some constants. It is easy to calculate that θ_+ satisfies the inequality $\mathbf{B}(\theta_+) \le 0$ a.e. in $\mathbb{R}_+$ if

$$C_1\xi^\gamma D^{-1}\left(\xi^{-\delta} - \xi_1^{-\delta}\right) + C_2\left(\xi_2^{-\delta} - \xi^{-\delta}\right) - A^{p-1}\xi^{\frac{p-1}{m}} D^{\frac{p-1}{m-1}} < 0 \tag{4.188}$$

for $\xi \in (0, c)$, where $D = c^\gamma - \xi^\gamma$, $\delta = (1+1/m) - \gamma > 0$ and

$$C_1 = \frac{m\gamma^2}{(m-1)^2}A^{m-1}, \quad C_2 = \frac{m\gamma(1+\gamma)}{m-1}A^{m-1},$$

$$\xi_1^{-\delta} = \frac{(p-m)(m-1)}{2m\gamma(p-1)A^{m-1}}, \quad \xi_2^{-\delta} = \frac{(p+m)(m-1)}{2m^2(p-1)\gamma(1+\gamma)A^{m-1}}.$$

Notice that $\xi_1 > \xi_2$.

Fix an arbitrary $A > 0$. Since $\gamma \in (0, 1+1/m)$ (i.e., $\delta > 0$), one can see that (4.188) is valid for any $c > 0$ large enough in a small neighbourhood $(0, \varepsilon)$, where the second negative term in the left-hand side of (4.188) is dominant. It is also easily seen that (4.188) is valid for any $\xi \in (\varepsilon, \xi_1)$ provided that c is large enough. Setting $\xi = \alpha c$, where $\alpha \in [\xi_1/c, 1)$, we arrive at the inequality which can be studied by the same technique as in the proof of Lemma 4.20. This implies that (4.188) holds on $[\xi_1, c]$, if c is large enough.

Thus, by comparison we have that for $\tau \gg 1$, there holds $\theta(\xi, \tau) \leq \theta_+(\xi)$ in $\mathbb{R}_+$, and choosing $c = c_+ > 0$ and $\alpha_+ > 0$ large enough yields the upper estimate in (4.181). □

It follows from the upper bound in (4.181) that the evolution orbit $\{\theta(\cdot, \tau), \tau > 0\}$ is compact in $C(\mathbb{R}_+)$, and in particular

Corollary 4.25 *We have*

$$|(\theta^{m-1})_\xi| \leq C \quad in\ \mathbb{R}_+ \times (1, \infty). \tag{4.189}$$

This estimate implies that the rescaled function $\theta(\xi, \tau)$ is also Hölder continuous in τ; see comments at the end of the chapter.

As a second step in the proof of Theorem 4.17, we establish the following stabilization result for monotone solutions.

Lemma 4.26 *Assume that $\theta(\xi, \tau)$ is nonincreasing (nondecreasing) in $\tau \in (0, \infty)$ for any $\xi \in \mathbb{R}_+$. Then there exists the limit*

$$\theta(\xi, \tau) \to f(\xi) \quad as\ \tau \to \infty, \tag{4.190}$$

and $f \not\equiv$ constant *solves the stationary problem* (4.137), (4.138).

Proof. Let for instance $\theta(\xi, \tau)$ be nonincreasing in time for $(\xi, \tau) \in \mathbb{R}_+ \times \mathbb{R}_+$. Since by Lemma 4.24 the function $\theta(\xi, \tau)$ is bounded from below, we conclude that the limit in (4.190) exists and $f(\xi) \geq \alpha_- F(\xi; c_-)$ for $\xi \geq 0$.

Let us now show that $f(\xi)$ solves (4.137), (4.138). Since by hypotheses $\theta_\tau \leq 0$ a.e., we deduce that the function

$$E(\tau) = \int_0^\infty [\theta(\xi, \tau) - f(\xi)]d\xi, \tag{4.191}$$

which plays a role of a Lyapunov function (monotone on such evolution orbits), satisfies $E(\tau) \to 0$ as $\tau \to \infty$. By using the fact that

$$E(\tau) \equiv -\int_\tau^\infty ds \int_0^\infty \theta_\tau(\xi, s)d\xi \equiv \int_\tau^\infty ds \int_0^\infty |\theta_\tau(\xi, s)|d\xi, \tag{4.192}$$

we have the estimate

$$\int_1^\infty \|\theta_\tau(\cdot, \mu)\|_{L^1(\mathbb{R}_+)} d\mu < \infty. \tag{4.193}$$

Fix an arbitrary monotone sequence $\{\tau_j\} \to \infty$. From (4.193) we have that uniformly in $s \in (0, 1)$,

$$0 \leq \int_0^\infty [\theta(\xi, \tau_j) - \theta(\xi, \tau_j + s)]d\xi = \int_{\tau_j}^{\tau_j + s} \|\theta_\tau(\cdot, \mu)\|_{L^1(\mathbb{R}_+)} d\mu$$

$$\leq \int_{\tau_j}^{\infty} \|\theta_\tau(\cdot, \mu)\|_{L^1(\mathbb{R}_+)} d\mu \to 0 \tag{4.194}$$

as $j \to \infty$. By passing to the limit $\tau = \tau_j + s \to \infty$ in equation (4.179), in view of the above estimates we deduce that $\theta(\cdot, \tau_j + s) \to f(\cdot, s)$ in $L^\infty_{\text{loc}}(\mathbb{R}_+ : C(\mathbb{R}_+))$, where $f(\cdot, s)$ is a weak solution to the equation $f_s = \mathbf{B}(f)$. From (4.194) we conclude that the function f does not depend on s. By known standard regularity properties of weak solutions to equations of the porous medium type, we then deduce that f is a nontrivial stationary solution. □

The next step is to identify the limit profile f. This is done via a uniqueness result.

Lemma 4.27 *A nonnegative compactly supported solution $f \not\equiv 0$ to the stationary problem* (4.137), (4.138) *is unique.*

Proof. We argue by contradiction. Assume that there exist two solutions, $f \not\equiv F$, of the problem (4.137), (4.138). Without loss of generality we suppose that

$$\{\xi \geq 0 : f(\xi) < F(\xi)\} \neq \emptyset. \tag{4.195}$$

For a fixed $\lambda > 1$, denote

$$f_\lambda(\xi) = \lambda^{2/(m-1)} f(\xi/\lambda). \tag{4.196}$$

Then we have

$$\mathbf{B}(f_\lambda) \equiv f^p \lambda^{2/(m-1)} (1 - \lambda^{2(p-1)/(m-1)}) < 0 \quad \text{if} \quad f(\xi/\lambda) > 0, \tag{4.197}$$

i.e., f_λ is a strict supersolution of equation (4.137). It follows from (4.195) and (4.196) that

$$\lambda_* = \inf\{\lambda > 1 : f_\lambda(\xi) \geq F(\xi) \quad \text{for all} \quad \xi > 0\} > 1. \tag{4.198}$$

Set $f_* = f_{\lambda_*}$. By the maximum principle, from (4.197) we conclude that

$$f_*(\xi) > F(\xi) \quad \text{inside the support } \operatorname{supp} F = [0, \xi_0]. \tag{4.199}$$

We now prove that

$$(f_*^m)'(0) > (F^m)'(0). \tag{4.200}$$

Indeed, integrating over $(0, \xi)$ both equation (4.137) for F and inequality (4.197) for f_* and subtracting, we arrive at the inequality

$$\begin{aligned}(f_*^m - F^m)' < (f_*^m)'(0) - (F^m)'(0) - \frac{m+2-p}{2(p-1)} \int_0^\xi (f_* - F) d\eta \\ - \frac{p-m}{2(p-1)} (f_* - F) + \int_0^\xi (f_*^p - F^p) d\eta.\end{aligned} \tag{4.201}$$

Assuming now that (4.200) is false and $(f_*^m)'(0) \le (F^m)'(0)$, we then conclude from (4.201) that $\left(f_*^m - F^m\right)' < 0$ for small $\xi > 0$. Integrating again over $(0, \xi)$ yields $f_*^m < F^m$ for small $\xi > 0$ contradicting (4.198).

Thus, from (4.198)–(4.200) we derive the last possibility

$$f_* > F \quad \text{on} \quad (0, \xi_0) \quad \text{and} \quad f_*(\xi_0) = F(\xi_0) = 0. \tag{4.202}$$

We now show that (4.202) is impossible. Consider a self-similar solution of the form (4.136),

$$u_F(x,t) = (1+t)^{-1/(p-1)} F(x/(1+t)^\gamma), \quad \gamma = (p-m)/2(p-1),$$

and the corresponding supersolutions of equation (4.110)

$$u_\lambda(x,t) = (1+t)^{-1/(p-1)} f_\lambda(x/(1+t)^\gamma) \quad (\lambda > 1).$$

Then (4.202) implies that $u_{\lambda_*} \ge u_F$ in Q. By continuity, it follows from (4.199) and (4.200) that there exists $\varepsilon > 0$ small enough such that $u_{\lambda_*}(x,\varepsilon) > u_F(x,0)$ in $(0, \xi_0]$, i.e., $(1+\varepsilon)^{-\frac{1}{p-1}} f_*(x/(1+\varepsilon)^\gamma) > F(x)$ in $(0, \xi_0]$. Since now $\operatorname{supp} f_*(x/(1+\varepsilon)^\gamma) = \xi_0(1+\varepsilon)^\gamma > \xi_0$, there exists $\lambda' \in (1, \lambda_*)$ such that

$$u_{\lambda'}(x,\varepsilon) \equiv (1+\varepsilon)^{-\frac{1}{p-1}} f_{\lambda'}(x/(1+\varepsilon)^\gamma) \ge F(x) \quad \text{for all } x \ge 0.$$

Then by comparison we have that $u_{\lambda'}(x, t+\varepsilon) \ge u_F(x,t)$ in Q and hence

$$(1+t+\varepsilon)^{-\frac{1}{p-1}} f_{\lambda'}(x/(1+t+\varepsilon)^\gamma) \ge (1+t)^{-\frac{1}{p-1}} F(x/(1+t)^\gamma). \tag{4.203}$$

Multiplying both sides of (4.203) by $(1+t)^{1/(p-1)}$ and introducing the new variable $\xi = x/(1+t)^\gamma$ yields $(1+O(t^{-1})) f_{\lambda'}(\xi(1+O(t^{-1}))) \ge F(\xi)$ for $t \gg 1$. Passing to the limit $t \to \infty$, we have $f_{\lambda'}(\xi) \ge F(\xi)$ for $\xi \ge 0$ with $\lambda' \in (1, \lambda_*)$, which contradicts (4.198) and completes the proof. □

Proof of Theorem 4.17. We can now end the proof. It follows from the proof of Lemma 4.24 that there exists $\tau_3 > 0$ large enough such that

$$\theta_-(\xi) \le \theta(\xi, \tau_3) \le \theta_+(\xi) \quad \text{for} \quad \xi > 0, \tag{4.204}$$

where θ_- (resp. θ_+) is the weak subsolution (supersolution) to (4.179), i.e.,

$$\mathbf{B}(\theta_-) \ge 0 \quad (\mathbf{B}(\theta_+) \le 0) \quad \text{a.e. in } \mathbb{R}_+. \tag{4.205}$$

For instance, we can take functions $\theta_\pm(\xi)$ given in the proof of Lemma 4.24. Denote by $\underline{\theta}(\xi,\tau)$ (resp. $\overline{\theta}(\xi,\tau)$) the solution to the problem (4.179), (4.180) with the initial function θ_- (resp. θ_+). Then by comparison

$$\underline{\theta}(\xi,\tau) \le \theta(\xi,\tau) \le \overline{\theta}(\xi,\tau) \quad \text{in } \mathbb{R}_+ \times (\tau_3, \infty). \tag{4.206}$$

By the maximum principle, inequalities (4.205) imply that

$$(\underline{\theta})_\tau \geq 0 \quad (\overline{\theta}_\tau \leq 0) \quad \text{a.e. in } \mathbb{R}_+ \times (\tau_3, \infty). \tag{4.207}$$

Then Lemmas 4.26 and 4.27 yield that

$$\underline{\theta}(\xi, \tau) \quad (\overline{\theta}(\xi, \tau)) \quad \to f(\xi) \quad \text{as } \tau \to \infty, \tag{4.208}$$

where $f \not\equiv 0$ is the unique compactly supported stationary solution. Hence, by (4.206) the convergence (4.208) holds for the solution $\theta(\xi, \tau)$, completing the proof. □

Supercritical case. Proof of Theorem 4.18. In the supercritical exponent range $p > m+1$ we consider the simple rescaling given by formula (4.140)

$$\theta(\xi, \tau) \equiv (1+t)^{1/m} u(\xi(1+t)^{1/2m}, t),$$

with $\tau = \ln(1+t)$. Then $\theta(\xi, \tau)$ solves the following quasilinear equation:

$$\theta_\tau = \mathbf{A}(\theta) - e^{-\alpha\tau}\theta^p \quad \text{in } Q, \tag{4.209}$$

where $\mathbf{A}$ is the stationary operator (4.120) and $\alpha = [p-(m+1)]/m > 0$. It also satisfies the boundary and initial conditions

$$\theta(0, \tau) = 0 \quad \text{for } \tau \geq 0, \tag{4.210}$$

$$\theta(\xi, 0) = \theta_0(\xi) \equiv u_0(\xi) \quad \text{for } \xi > 0. \tag{4.211}$$

As above, we begin with some sharp upper and lower estimates.

Lemma 4.28 *Let $p > m+1$. Then there exist positive constants $c_+ > c_-$ such that for all $\tau \gg 1$,*

$$F(\xi; c_-) \leq \theta(\xi, \tau) \leq F(\xi; c_+) \quad in \ \mathbb{R}_+. \tag{4.212}$$

Proof. It follows from equation (4.209) and (4.120) that the function $\theta_+(\xi) = F(\xi; c_+)$ is a weak supersolution to (4.209) for arbitrary $c_+ > 0$, and the upper estimate in (4.212) follows by comparison.

Such a weak subsolution does not exist, and we consider a perturbed function

$$\theta_-(\xi, \tau) = \frac{g(t)}{\phi^{1/m}(t)} F(\xi\phi(t); c_-), \tag{4.213}$$

where smooth positive functions g, ϕ will be determined below. Then θ_- satisfies

$$(\theta_-)_\tau \leq \mathbf{A}(\theta_-) - (1+t)^{-\alpha}(\theta_-)^p \qquad \text{a.e. in } Q \quad \text{if}$$

$$D^{-1}\xi^{\frac{m+1}{m}}\left[(1+t)\frac{m+1}{m(m-1)}\phi^{\frac{1}{m}}\phi' - \frac{m+1}{2m^2(m-1)}\phi^{\frac{m+1}{m}}\left(1 - g^{m-1}\phi^{\frac{m+1}{m}}\right)\right]$$
$$+\left\{\frac{2m+1}{2m^2}\left[1 - g^{m-1}\phi^{\frac{m+1}{m}}\right] - (1+t)\frac{g'}{g}\right\}$$
$$- (1+t)^{-\alpha}(A_0 g)^{p-1}\xi^{\frac{p-1}{m}} D^{\frac{p-1}{m-1}} \geq 0,$$

where $D = c_-^{(m+1)/m} - (\phi\xi)^{(m+1)/m}$. Setting

$$1 - (g(t))^{m-1}(\phi(t))^{\frac{m+1}{m}} \equiv 0 \tag{4.214}$$

yields the inequality

$$D^{-1}\xi^{\frac{m+1}{m}}\left[(1+t)\frac{m+1}{m(m-1)}\phi^{\frac{1}{m}}\phi'\right] - (1+t)\frac{g'}{g}$$

$$-(1+t)^{-\alpha}(A_0 g)^{p-1}\xi^{\frac{p-1}{m}}D^{\frac{p-1}{m-1}} \geq 0. \tag{4.215}$$

Let

$$g(t) = 1 + a(1+t)^{-\alpha}, \tag{4.216}$$

where $a > 0$ is a constant. Then (4.215) has the form

$$D^{-1}\xi^{\frac{m+1}{m}}\frac{\alpha a}{g^m} + \frac{\alpha a}{g} - (A_0 g)^{p-1}\xi^{\frac{p-1}{m}}D^{\frac{p-1}{m-1}} \geq 0,$$

and hence it is valid if $\alpha a - A_0^{p-1}(g(t))^p \sup_{\xi>0, D>0} \xi^{\frac{p-1}{m}} D^{\frac{p-1}{m-1}} \geq 0$ for $t > 0$. This yields

$$c_-^{\frac{2(p-1)}{m-1}} \leq \alpha a A_0^{1-p}\left(\frac{m-1}{2m}\right)^{-\frac{p-1}{m+1}}\left(\frac{m+1}{2m}\right)^{-\frac{p-1}{m-1}}(1+a)^{-\frac{2mp-(m-1)}{m+1}}. \tag{4.217}$$

By choosing constants $a > 0$ and $c_- > 0$ satisfying (4.217) and by using the space-time structure of the subsolution (4.213), (4.214), (4.216), we deduce that the lower estimate in (4.212) holds for large τ provided that $c_- > 0$ is small enough. This completes the proof of Lemma 4.28. □

Proof of Theorem 4.18. Next, we let $\tau \to \infty$. Since by regularity the passage to the limit $\tau = \tau_j + s \to \infty$ in (4.209) yields the equation $w_s = \mathbf{A}(w)$ with the autonomous operator (4.120) arising in the critical case $p = m + 1$, the proof of (4.177) and (4.178) is quite similar to that given above.

Finally, having property (4.178), the uniqueness of the asymptotic profile corresponding to the unique value of the constant $c_\infty > 0$ in (4.140) is a straightforward consequence of the inequality for the momentum

$$M(\tau) = \int_0^\infty \xi\theta(\xi,\tau)\,d\xi \equiv \int_0^\infty xu(x,t)\,dx, \tag{4.218}$$

which reads

$$\frac{dM}{d\tau} = -e^{-\alpha\tau}\int_0^\infty \xi\theta^p(\xi,\tau)d\xi < 0 \qquad \text{for } \tau > 0. \tag{4.219}$$

This follows from equation (4.209) by integration. Then there exists the limit

$$M(\tau) \to M_\infty \geq 0 \quad \text{as } \tau \to \infty, \tag{4.220}$$

where the constant M_∞ depends on the initial data. By the lower bound in (4.212) M_∞ is positive. Moreover, if $f \in \omega(\theta_0)$, then by (4.177), (4.178) we have $f(\cdot) \equiv F(\cdot; c_\infty)$ for some $c_\infty \in [c_-, c_+]$, and therefore by (4.219) the constant $c_\infty > 0$ is uniquely determined from the equation (see (4.122) with $M = M_\infty$)

$$\int_0^\infty \xi F(\xi; c_\infty) d\xi = M_\infty. \tag{4.221}$$

This completes the proof. □

Interface behaviour. There is an interesting aspect of our problem that we have not dealt with in previous sections, namely the asymptotic behaviour of the *interface* which bounds the support of a solution. Indeed, it is well known that for $m > 1$ equation (4.110) has the property of finite propagation, whereby compactly supported data give rise to solutions with compact support in the space variable for every fixed $t > 0$. This property is not true if $m = 1$, $p > 1$. Therefore, assuming $p > m > 1$, we define

$$s(t) = \sup\{x > 0 : u(x, t) > 0\}. \tag{4.222}$$

It then follows that $s : [0, \infty) \to [0, \infty)$ is continuous and nondecreasing. Moreover, it goes to infinity as $t \to \infty$. The curve $x = s(t)$ is called the outer interface (or free boundary).

Our previous analysis not only gives the exact rate of decay of the solutions and the limiting profiles to which they tend, but we can also obtain the precise growth as $t \to \infty$ of the interface. Since the proofs do not need more than the techniques used in the Cauchy problem, we feel justified in stating only the results for the reader's convenience or future reference. In fact, the proof of convergence of interfaces is a straightforward consequence of the convergence of profiles and some extra results on the nonexistence for large times of a small "tail" of the solution. As $t \to \infty$, we have:

(i) If $m < p < m + 1$, then $s(t) \approx c_* t^\nu$, $\nu = (p - m)/2(p - 1)$, where c_* is the endpoint of the support of the profile f of Theorem 4.17.

(ii) If $p = m + 1$, then $s(t) \approx c_*[t (\ln t)^{-(m-1)}]^\nu$, $\nu = 1/2m$. Here c_* is the constant given in (4.125).

(iii) If $p > m + 1$, then $s(t) \approx c_\infty t^\nu$, $\nu = 1/2m$, where c_∞ is given in Theorem 4.18. This is the only case where the constant depends on the initial data.

Comparison with the rates of the Cauchy problem could be interesting. In the critical and subcritical cases we have the same growth exponents with larger universal constants (but remark that here critical means $p = m + 2$, larger than $m + 1$). In the supercritical case the exponent is $\nu = 1/(m + 1)$. Clearly, the growth rates of the Cauchy problem are in all cases equal or larger (in fact larger) than those of the present mixed problem.

General nonlinearities. Let us now discuss some generalizations of our results to equation $u_t = (u^m)_{xx} - f(u)$, having a more general absorption term.

(i) First, all the results are true if the function $f(u)$ satisfies

$$f(u)/u^p \to C \quad \text{as } u \to 0, \tag{4.223}$$

where $C > 0$ is an arbitrary constant, and the adaptation is immediate. Indeed, by rescaling we can put $C = 1$. Next, for given $\varepsilon > 0$, there holds

$$1 - \varepsilon \leq f(u)/u^p \leq 1 + \varepsilon \tag{4.224}$$

for all small $u > 0$. Therefore by comparison, for large $t \gg 1$, we bound the solution $u(x,t)$ from above and below by the supersolution $\overline{u}_\varepsilon(x,t)$ and the subsolution $\underline{u}_\varepsilon(x,t)$ satisfying the equation with the absorption term replaced by $(1-\varepsilon)u^p$ and $(1+\varepsilon)u^p$ respectively (which are reduced to the standard equation (4.110) by rescaling). Then applying Theorems 4.15 – 4.18 to $\overline{u}$ and $\underline{u}$ and passing to the limit $\varepsilon \to 0$, we arrive at the same asymptotic behaviour for the solution u.

(ii) For the supercritical case $p > m + 1$ it is clear that the asymptotic behaviour given in Theorem 4.18 is true for any f satisfying

$$f(u)/u^p \leq C \tag{4.225}$$

for small $u > 0$.

(iii) In the subcritical case $p \in (m, m+1)$ since the absorption term is strongly involved in the asymptotic behaviour, we cannot expect the behaviour to be the same in all detail under a rough condition like (4.225). Nevertheless, one can check by the methods mentioned above that under the estimates

$$C_1 \leq f(u)/u^p \leq C_2 \tag{4.226}$$

($C_1 < C_2$ are positive constants) both the rate of decay and the size of the support of the solution for $t \gg 1$ are given by the self-similar spatio-temporal structure of (4.136),

$$\sup_x u(x,t) \sim t^{-1/(p-1)}, \qquad \text{meas}\,(\text{supp}\, u(x,t)) \sim t^{(p-m)/2(p-1)}.$$

Similarly, in the critical case $p = m+1$ the assumptions (4.226) imply the asymptotic rescaling of the amplitude and the size of the support given by (4.123).

Finally, we can consider also nonpower diffusion nonlinearities as in equation (4.116). Under the assumptions (4.223) and $\lim_{u\to 0} \phi(u)/u^m = C_1$ we are dealing with an asymptotically small perturbation of equation (4.110), so that the classification and asymptotic behaviour of each case still hold. However, in this case the technical details are not so immediate and we will not go into more detail here, directing the reader to the papers [202] and [169] for references to the available techniques to deal with such problems.

Remarks and comments on the literature

In Sections 4.1–4.9 we mainly follow the results of [169], 1991, announced in [311]. Much of the background material can be found in our Chapter 2. Here are some further comments.

§ 4.1, 4.2. The *instantaneous source-type* solution (4.4) of the PME $u_t = \Delta u^{1+\sigma}$ was first constructed in [325] for $N = 1, 3$ and generalized to arbitrary dimensions in [25]. General existence and regularity results for continuous weak solutions of such quasilinear parabolic equations are available in [97], [95], [96], [202], [219], [304]. See [202] for a full list of references.

Let us present references on the noncritical, $\beta \neq \beta_*$, asymptotic behaviour. For $\beta > \beta_*$, the absorption term $-u^\beta$ is negligible for $t \gg 1$, see first results in [149], [186], [205], [206], [209] and Chapter 2 in [286]. For $\sigma + 1 < \beta < \beta_*$ the solution converges to the very singular solution of (4.1). For $m = 1$ the VSS was constructed in [149] by a PDE approach, where its stability properties are also established. An alternative ODE proof of existence of the VSS was given in [56]. For $m > 1$ the VSS was constructed in [265]. Uniqueness was settled in [213]. See a full list of references in [150], Chapter 2.

The critical case $\beta = \beta_*$ was first investigated in [149] for $\sigma = 0$, the semilinear heat equation $u_t = \Delta u - u^{1+2/N}$, $N \geq 1$ (a one-sided bound was earlier proved in [186]). The case $\sigma > 0$, $N = 1$, was first studied in [156] and [161] by using a different approach based on the construction of an approximate Lyapunov function which is "almost" monotone on the evolution orbits. For $N > 1$ such a function does not exist. We take from [156], [161] the super- and subsolutions and the rescaled mass analysis in the uniqueness proof.

In [57], [58], for semilinear equations with $\sigma = 0$ such logarithmically perturbed asymptotics were justified by a perturbation analysis of a linearized second-order self-adjoint operator and were shown to exist for a wide class of second-order semilinear evolution equations. It is worth mentioning that linearization techniques are not straightforward for quasilinear equations with $\sigma > 0$. A linearization procedure about compactly supported profiles like (4.5) or (4.66) even for $N = 1$ leads to a singular second-order symmetric ordinary differential operator on bounded intervals (unlike the semilinear case $\sigma = 0$) having singularities at finite end-points. Spectral properties, completeness of eigenfunctions for suitable self-adjoint extensions of such operators and especially applicability of such eigenfunction expansions for quasilinear PDEs are not known. For $N > 1$ these lead to hard problems on self-adjoint extensions of singular elliptic operators.

Another type of asymptotics with logarithmic contraction occurred in [209] in the case $\beta > \beta_*$ for solutions whose initial data are not compactly supported, but behave like $|x|^{-N}$ as $|x| \to \infty$. The authors call these contracted profiles *reconstructed* similarity solutions.

For completeness of the classification of the solutions relative to parameters β, σ, we note that for $\beta \in (1, 1 + \sigma)$ the solutions are localized and supports are uniformly bounded [202]. If $\beta = 1 + \sigma$ (a *critical* exponent), then supports expand logarithmically, and the diameter of the support behaves like $O(\ln t)$ for $t \gg 1$ and the asymptotic behaviour can be described by explicit non–self-similar solutions on a linear subspace invariant under quadratic operators, see [286], p. 105 and references therein.

A general theory of approximate self-similar solutions to one-dimensional heat conduction equations is presented in [166] and [286], Chapt. 6.

The critical absorption exponent $\beta_* = \sigma + 1 + 2/N$ has a more universal nature. The same critical Fujita exponent, first obtained in 1966 [125] for the semilinear equation with $\sigma = 0$, occurs for the equation with a source term

$$u_t = \Delta u^{\sigma+1} + u^\beta,$$

so that for $\beta \in (1, \beta_*]$ all the solutions $u \not\equiv 0$ blow-up in finite time while for $\beta > \beta_*$ there exists a class of small global-in-time solutions [147], [133], see the references in Chapt. 4 in [286] and the survey [239]. See also some comments on dipole-like behaviour studied in the last section. More precisely, at $\beta = \beta_*$ the trivial stationary solutions $u \equiv 0$ loses its stability: for $\beta > \beta_*$ it has a domain of attraction, while for $\beta \leq \beta_*$ it is unstable.

§ 4.3 – § 4.5. Lemmas 4.3 and 4.5 are proved in the previous works [156], [161]. These references also contain a study of the mass equation and the uniqueness of the stable asymptotics

§ 4.4. For contraction properties of the PME semigroup in $L^1(\mathbb{R}^N)$, see [40] and [211]. Global stability of the ZKB solutions of a fixed mass is proved in [123], see also preceding formal analysis in [5], [323]. See more information in Chapter 2.

§ 4.6. General results on existence and interior regularity for equations with the p-Laplacian operator can be found in [202] and [210], see also references therein. Lemma 4.11 was proved in [156]. Compactness of the rescaled orbit follows from general results in [97], [95], [96], [202]. Contractivity results for the p-Laplacian operator and other related properties are well known, cf. [210].

§ 4.7. Nonexistence of a solution $u \in L^\beta_{\text{loc}}((0, T) \times \mathbb{R}^N)$ of (4.1), $\beta \geq \beta_*$, with initial Dirac mass $u(x, 0) = \delta(x)$ was first proved in [55] in the semilinear case $\sigma = 0$ and in [207] for $\sigma > 0$. It is important that Theorem 4.12 establishes that any bounded approximation $\{u_n\}$ of solution $u = \lim u_n$ gives the trivial one $u \equiv 0$ for any $t > 0$. Moreover, estimate (4.88) describes the sharp rate of convergence of approximation $O((\ln n)^{-k}) \to 0$ as $n \to \infty$. This shows that $u \equiv 0$ is actually a unique solution of the problem, which however cannot satisfy the initial condition. This is a typical situation when an evolution semigroup of proper (or maximal, minimal or other type of) solutions to nonlinear PDEs constructed by approximation/truncation techniques is essentially *discontinuous* at $t = 0$. We say that it has an initial layer of discontinuity. There are several types of different complete/incomplete blow-up or extinction singularities where the discontinuities occur at finite times $t = T > 0$; see [174] and a general approach to extended semigroup theory in [177].

Other examples of disappearance after an initial layer are known in Nonlinear Diffusion. Thus, [107] describes this phenomenon for the nonnegative solutions of the diffusion equation $u_t = (u^\sigma u_x)_x$ in the singular range $-2 < \sigma \le -1$. In that case for *every* initial data $u_0 \in L^1(\mathbb{R})$ the solution $u_\sigma(x, t)$ defined in $\mathbb{R} \times (0, \infty)$ disappears as $\sigma \to -2$ in every set of the form $\mathbb{R} \times (t_0, \infty)$.

§ **4.8.** Estimate (4.89) for the PME is well known, cf. Property 5 of Section 2.2, or the original papers [16], [35], [316].

§ **4.9.** Existence and uniqueness results for equations like (4.102) can be found in [201], [202], [219]. See also Chapter 2.

§ **4.10.** The main results of this section are taken from our work [165], 1995. The dipole behaviour in the nonabsorption case has been studied in [211]. Localization results for $1 < p < m$ were first proved by A.S. Kalashnikov [201], see references in [202], see also [48]. Extra regularity results can be found in [20]. The fact that (4.189) implies Hölder continuity in τ is due to S.N. Kruzhkov [226], see a detailed proof in [20], Lemma 1.4. In the proof of Lemma 4.27 we follow [212]. Subsolution (4.213) is taken from [286], p. 236.

The dipole-like behaviour for critical and nearby parameters exhibits clear counterparts for several nonlinear heat equations including the equation with the p-Laplace operator and absorption

$$u_t = (|u_x|^\sigma u_x)_x - u^\beta, \quad \sigma > 0, \ \beta > \sigma + 1, \tag{4.227}$$

posed in $\mathbb{R}_+ \times \mathbb{R}_+$, with conditions (4.111), (4.112) as above. Such results and some extensions, which follow from application of the techniques already developed, show that we are dealing with a rather general phenomenon occurring for a wide class of equations and settings in nonlinear heat propagation.

It is interesting to note that self-similar "dipole-like" solutions of the purely p-Laplacian equation $u_t = (|u_x|^\sigma u_x)_x, \ x > 0, \ t > 0$, belong to the self-similarity of the *second kind*, a term introduced by Zel'dovich in 1956 [322], meaning that the self-similar exponents are obtained by solving a nonlinear eigenvalue problem, and not from dimensional considerations. The critical absorption exponent cannot be calculated explicitly. The phenomenon has been investigated in detail in [45]. See more details on self-similarities of second kind in comments to Chapter 6. The same is true for the dual PME with absorption

$$u_t = |u_{xx}|^{m-1} u_{xx} - u^\beta, \quad m > 1, \ \beta > 1 \quad (u \ge 0).$$

The critical Fujita exponent for the dual PME with the source term $+u^\beta$ was calculated in [151], and it is known to coincide with the corresponding critical absorption exponent for the above equation with absorption.